UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2009 9:00 am to 12:00 pm

PAPER 44

QUANTUM FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A real scalar field, $\phi(x)$, has a Lagrangian density

$$\mathcal{L}(x) = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m^2 \phi^2 \,.$$

Explain how to quantise the field theory and show that the Heisenberg field satisfies

$$(\partial^2 + m^2)\phi = 0\,.$$

Deduce that the field can be expressed in terms of mode operators, $a(p)$, in the form

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a(p)\, \mathrm{e}^{-ip\cdot x} + a^\dagger(p)\, \mathrm{e}^{ip\cdot x} \right) ,$$

where $p = (E_p, \mathbf{p})$ and $E_p = \sqrt{\mathbf{p}^2 + m^2}$. Show that the equal time canonical commutation relations imply

$$\left[a(p), a^\dagger(p') \right] = (2\pi)^3 \delta^{(3)} \left(\mathbf{p} - \mathbf{p}' \right) .$$

Express the Hamiltonian in terms of mode operators and show that

$$[H, a(p)] = -E_p a(p) \ ; \quad \left[H, a^\dagger(p) \right] = E_p a^\dagger(p)\,.$$

Hence explain the particle interpretation of the theory.

The Feynman propagator is defined as

$$D_F(x-y) = \langle\, 0 \,|\, T\phi(x)\phi(y) \,|\, 0 \,\rangle\,.$$

Show that

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}\, \mathrm{e}^{-ip\cdot(x-y)} \,,$$

and hence that

$$\left(\partial^2 + m^2\right) D_F(x-y) = -i\, \delta^{(4)}(x-y)\,.$$

2 The Dirac equation for a particle of mass m is

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi = 0$$

where $\psi(x)$ is the spinor wavefunction of the particle and the matrices, γ^{μ}, are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where 1 is a 2×2 unit matrix and σ^i are the Pauli matrices. Verify that

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}.$$

Show that $\psi(x)$ satisfies the Klein-Gordon equation for a relativistic particle of mass m.

For a Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ there exists a 4×4 matrix, $S(\Lambda)$, such that

$$S^{-1}(\Lambda)\,\gamma^{\mu} S(\Lambda) = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}.$$

Show that the Dirac equation is invariant under Lorentz transformations.

Show that the Dirac equation is invariant under the parity transformation

$$\psi(x) \;\rightarrow\; \psi_p(x) = \gamma^0 \psi(x_p)\,,$$

where $x \rightarrow x_p = (x^0, -\mathbf{x})$. Deduce the transformation of the Dirac conjugate spinor, $\overline{\psi}(x)$, under parity and the parity transformation of the bilinears $\overline{\psi}(x)\psi(x)$ and $\overline{\psi}(x)\gamma^5\psi(x)$, where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Discuss the importance of the projectors

$$P_{\pm} = \tfrac{1}{2}\,(1 \pm \gamma_5)$$

for the case of the massless Dirac equation.

3 A real scalar field has Lagrangian density

$$\mathcal{L} \;=\; \tfrac{1}{2}\,(\partial_\mu \phi)^2 \,-\, \tfrac{1}{2}\,m^2\phi^2 \;-\; \frac{\lambda}{4!}\,\phi^4\,.$$

Derive the Hamiltonian density, H, for this theory explaining how it can be split as

$$H \;=\; H_0 \,+\, H_I\,,$$

where H_0 is the Hamiltonian for a non-interacting theory of mass m and H_I the interaction Hamiltonian.

Explain how the interaction picture is used to obtain the result

$$S \;=\; T\exp\left\{-i\!\int d^4x\, H_I(x)\right\}$$

for the operator S which describes the transition of the system from the far past to the far future.

Two particles with 4-momentum p_1 and p_2 interact with final 4-momentum p_3 and p_4. Show that

$$\langle\, p_3, p_4\,|\,(S-I)\,|\,p_1, p_2\,\rangle \;=\; i\,(2\pi)^4\,\delta^{(4)}\,(p_3+p_4-p_1-p_2)\,A\,.$$

What is A to $\mathcal{O}(\lambda)$?

4 Consider the process

$$e^+ + e^- \;\to\; \gamma + \gamma\;.$$

Draw the leading order Feynman diagrams that contribute to this process. Write down the necessary Feynman rules and use these to write down the corresponding scattering amplitudes.

What constraint, if any, is there on the polarisation vector of the photon with momentum k^μ? Show that the scattering amplitude is unaffected if ϵ^μ is replaced by $(\epsilon+\alpha k)^\mu$, where α is a real constant. What is the physical interpretation of this result?

Now consider electron photon scattering, $e^- + \gamma \to e^- + \gamma$. Write down the diagrams and scattering amplitudes to $\mathcal{O}(e^2)$ for this process. How does this differ from electron positron annihilation?

END OF PAPER