UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 29 May, 2009 1:30 pm to 4:30 pm

PAPER 43

STATISTICAL FIELD THEORY AND APPLICATIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Statistical Field Theory

Explain briefly what is meant by a phase diagram. Give an example of a three-dimensional phase diagram which contains a tri-critical point and describe the nature of the transitions which occur.

Give an account of the Landau-Ginsberg theory of phase transitions in the context of a scalar field theory paying particular attention to the following topics:

(a) The idea of an *order parameter*;

(b) The definition and relevance of the *correlation length*;

(c) The idea of universality;

(d) The occurrence of first-order and continuous phase transitions and how their defining properties are explained;

(e) The prediction of tri-critical behaviour and the conditions under which it occurs;

(f) The idea of a *critical exponent* and how they are predicted.

The Ising model in D dimensions is defined on a cubic lattice Λ with N sites and with spin $\sigma_{\mathbf{n}} \in \{1, -1\}$ on the site with position $\mathbf{n}$. The Hamiltonian with zero magnetic field is

$$\mathcal{H}(\{\sigma\}) = -J \sum_{\mathbf{n}\in\Lambda,\mu} \sigma_{\mathbf{n}}\sigma_{\mathbf{n}+\mu} \ ,$$

where the sum on μ is over the basis vectors of the lattice Λ. Using the mean-field approximation show that the magnetization M satisfies the equation

$$M = \tanh\left(2DJM/k_B T\right) \ ,$$

where k_B is Boltzmann's constant and T is temperature. Hence, show that in this case mean-field theory predicts a continuous phase transition, and determine the critical temperature T_C. By establishing a power series expansion in M for the 'tanh' function on the RHS of this equation show that, for small $\tau = (T_C - T)/T_C \ \ \tau > 0$,

$$M^2 = 3\tau \ + \ O(\tau^2) \ .$$

What does this result imply for the critical exponent β predicted by mean-field theory.

2 Statistical Field Theory

A spin model in D dimensions is defined on a lattice of spacing a with N sites and with spin σ_r on the r-th site. The Hamiltonian is defined in terms of set of operators $O_i(\{\sigma_r\})$ by

$$\mathcal{H}(\mathbf{u}, \sigma) = \sum_i u_i O_i(\{\sigma\}) \ ,$$

where u_i are coupling constants with $\mathbf{u} = (u_1, u_2, \ldots)$. The partition function is given by

$$\mathcal{Z}(\mathbf{u}, C, N) = \sum_\sigma \exp(-\beta\mathcal{H}(\mathbf{u}, \sigma) - \beta NC) \ .$$

Define the two-point correlation function $G(\mathbf{r})$ for the theory and state how the correlation length ξ parameterizes its behaviour for $|\mathbf{r}| \gg \xi$. State how the susceptibility χ can be expressed in terms of $G(\mathbf{r})$.

Explain how a renormalization group (RG) transformation may be defined in terms of a blocking kernel and state how a and N rescale in terms of the RG scale factor b.

The values of the parameters $(\mathbf{u}, C)$ after p blockings are denoted $(\mathbf{u}_p, C_p)$. The RG transformation for $\mathbf{u}$ can be written as

$$\mathbf{u}_p \to \mathbf{u}_{p+1} = \mathbf{R}(\mathbf{u}_p) \ .$$

Write down the form of the corresponding RG transformation for $C_p \to C_{p+1}$. What is the physical interpretation of C_p?

Derive the *RG* equation for the free energy $F(\mathbf{u}_p, C_p)$ and explain how it may be expressed in terms of a singular part $f(\mathbf{u})$ which obeys the RG equation

$$f(\mathbf{u}_0) = b^{-pD} f(\mathbf{u}_p) + \sum_{j=0}^{p-1} b^{-jD} g(\mathbf{u}_j) \ .$$

What is the origin of the function $g(\mathbf{u})$ which determines the inhomogeneous part of this transformation?

Interpret the behaviour of the RG equations in the neighbourhood of a fixed point in terms of the rescaling of the couplings $\mathbf{u}$ and of the spins $\{\sigma\}$. Explain briefly the concept of *relevant* and *irrelevant* operators.

By considering the behaviour of $f(\mathbf{u}_p)$ in the neighbourhood of the fixed point explain how the critical exponents of the continuous phase transition associated with that fixed point may be calculated, and briefly state the assumptions which permit the inhomogeneous part of the RG transformation to be neglected.

In the case that there are two relevant couplings $t = (T - T_C)/T_C$ and magnetic field h, derive the scaling hypothesis for the singular part F_s of the free energy

$$F_s = |t|^{D/\lambda_t} f_\pm\left(\frac{h}{|t|^{\lambda_h/\lambda_t}}\right) \ ,$$

where the meaning of the "$\pm$" label should be given.

The following critical exponents ν, γ, α are defined for $h = 0$:

$$\xi \;\sim\; |t|^{-\nu}, \qquad \chi \;\sim\; |t|^{-\gamma}, \qquad C_V \;\sim\; |t|^{-\alpha}\,,$$

where χ is the susceptibility and C_V is the specific heat at constant volume. Derive the scaling relation $\alpha = 2 - D\nu$.

According to the scaling hypothesis the correlation function takes the form

$$G(\mathbf{r}) = \begin{cases} \dfrac{1}{|\mathbf{r}|^{D-2+\eta}}\, f_G(|\mathbf{r}|/\xi) & |\mathbf{r}| \ll \xi, \\ \dfrac{\xi}{(\xi|\mathbf{r}|)^{(D-1)/2}}\, \exp(-|\mathbf{r}|/\xi) & |\mathbf{r}| \gg \xi. \end{cases}$$

From this parametrization obtain an expression for the susceptibility and derive the identity $\gamma = (2 - \eta)\nu$.

3 Statistical Field Theory and Applications

Consider the Gaussian model in $2 < D < 4$ dimensions, with Hamiltonian

$$H_0[\phi] \;=\; \int \mathrm{d}^D x \left[\frac{1}{2}\alpha^{-1}|\nabla\phi|^2 \;+\; \frac{1}{2}m^2\phi^2\right] \tag{1}$$

and $\phi(\mathbf{x})$ a real-valued field.

1. Find the Hamiltonian governing the Fourier transformed fields $\tilde{\phi}(\mathbf{p})$.

2. Introduce an external field $\tilde{J}$ which couples to the $\tilde{\phi}$ field, and show that the partition function in the presence of this field can be written as

$$Z_0[\tilde{J}] \;=\; Z_G \exp\left[\frac{1}{2}\int \frac{\mathrm{d}^D p}{(2\pi)^D}\,\tilde{J}\tilde{\Delta}^{-1}\tilde{J}^*\right] \tag{2}$$

(or something similar depending on your convention for how $\tilde{J}$ is introduced). Give explicit expressions for $\tilde{\Delta}(\mathbf{p})$ and Z_G.

3. Use $Z_0[\tilde{J}]$ to find the connected 2-point function $\tilde{G}_0(\mathbf{p})$ in the $\tilde{J} \to 0$ limit. Show that the asymptotic behaviour of the connected 2-point function in position space is

$$G_0(r) \sim \begin{cases} \dfrac{1}{r^{D-2}} & r \ll \xi \\[2ex] \dfrac{\xi e^{-r/\xi}}{(r\xi)^{(D-1)/2}} & r \gg \xi\,. \end{cases} \tag{3}$$

where $\xi^{-2} = \alpha m^2$.

4. Let the external source be a constant in position $J(x) = h$. Perform a renormalisation group transformation and determine how the coefficients α, m^2, and h transform.

5. How is critical behaviour (i.e. a phase transition) apparent in this analysis? Determine the critical exponent ν which governs the divergence of the correlation length.

4 Statistical Field Theory and Applications

Consider the Ising model and the nonlinear σ model in a low number of dimensions.

1. Give a heuristic argument for why a phase transition occurs in the two-dimensional Ising model, but not the one-dimensional Ising model.

2. Consider the nonlinear σ model in $D = 2 + \epsilon$ dimensions ($\epsilon \geqslant 0$), with Hamiltonian

$$\frac{H}{T} \;=\; \int \mathrm{d}^D x \, \frac{1}{2g} \, |\nabla \sigma|^2 \tag{1}$$

where σ is an n-component field with unit norm.

(a) Show that the RG transformation for the coupling g obeys

$$\frac{\mathrm{d}g}{\mathrm{d}\log b} \;=\; -\epsilon g \;+\; \frac{S_D}{(2\pi)^D} \Lambda^\epsilon (n-2) g^2 \tag{2}$$

upon integrating over wavevectors in the interval $(\Lambda/b, \Lambda)$. (S_D is the surface area of a unit sphere in D dimensions.)

(b) Find the fixed points g_* of the RG transformation.

(c) Find the thermal eigenvalue y_T (recall $g \propto T$), which is obtained by looking at the flow near g_*

$$\frac{\mathrm{d}}{\mathrm{d}\log b}(g_* + \delta g) \;\equiv\; y_T \, \delta g \,. \tag{3}$$

END OF PAPER