UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June 2009 9:00 am to 11:00 am

PAPER 42

INTRODUCTION TO SUPERSYMMETRY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A 3-dimensional supersymmetric quantum mechanical system has supercharge

$$Q = \sum_a \left(\psi_a A_a^\dagger + \psi_a^\dagger A_a \right)$$

where $A_a = \partial_a - \partial_a \chi$, with $\chi(\mathbf{x})$ a potential function, and

$$\left\{ \psi_a, \psi_b \right\} = \left\{ \psi_a^\dagger, \psi_b^\dagger \right\} = 0\,, \ \left\{ \psi_a, \psi_b^\dagger \right\} = \delta_{ab}\, \mathbf{1}\,.$$

The Hamiltonian $H = Q^2$ is

$$H = (-\Delta + \partial_a\chi\partial_a\chi + \Delta\chi)\,\mathbf{1} - 2\sum_{a,b} \partial_a\partial_b\chi\, \psi_a^\dagger \psi_b\,,$$

where $\Delta = \partial_a\partial_a$. Indices a, b run from 1 to 3.

Explain how the Hilbert space of states splits into four sectors. Describe the general structure of the spectrum of H in these sectors, and how states are related by the action of Q. Express, in the simplest way that you can, the form of the Hamiltonian H restricted to the two sectors where it reduces to a scalar operator.

Assume now that χ is a function only of the radial coordinate $r = (x^a x^a)^{\frac{1}{2}}$. Suppose there is a positive energy state in the sector $\mathcal{H}_0$ of Hilbert space annihilated by the operators ψ_a, whose wavefunction depends only on r. Construct the state related to this one by supersymmetry, and express it in its simplest form. If there is a zero energy state in $\mathcal{H}_0$, what is its wavefunction?

2 In a variant of the supersymmetric classical mechanics of a particle moving in one dimension, the Lagrangian is

$$L = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}U(x)^2 - \tfrac{1}{2}\psi_1\dot{\psi}_1 + \tfrac{1}{2}\psi_2\dot{\psi}_2 + U'(x)\psi_1\psi_2$$

where $x(t)$ takes values in the even part of a Grassmann algebra B, and $\psi_1(t)$ and $\psi_2(t)$ take values in the odd part. $U(x)$ may be regarded as a polynomial in x with real coefficients.

Find the equations of motion for x, ψ_1 and ψ_2, and show that the energy

$$E = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}U(x)^2 - U'(x)\psi_1\psi_2$$

and the supercharge

$$Q = \dot{x}\psi_1 - U(x)\psi_2$$

are conserved. Find a further, independent, conserved supercharge.

Assume now that $\dot{x} = U(x)$ and $\psi_1 = -\psi_2$. Show that the equations of motion are satisfied, provided

$$\dot{\psi}_1 = -U'(x)\psi_1\,.$$

For $U(x) = 1 + cx$, find the solution $\{x(t), \psi_1(t)\}$ of this restricted type, in terms of initial data $\{x(0), \psi_1(0)\}$. What are the values of the energy and the supercharges for this solution?

3 In the superspace extension of 1+1 dimensional Minkowski space, the supersymmetry operators are

$$Q_+ = \frac{\partial}{\partial\theta_+} + i\theta_+\partial_+\ , \quad Q_- = \frac{\partial}{\partial\theta_-} + i\theta_-\partial_-\ ,$$

and the field equation for a scalar superfield Φ can be written as

$$\left(i\frac{\partial^2}{\partial\theta_-\partial\theta_+} \ + \ \theta_-\frac{\partial}{\partial\theta_+}\partial_- \ - \ \theta_+\frac{\partial}{\partial\theta_-}\partial_+ \ - \ i\theta_-\theta_+\partial_-\partial_+ \right)\Phi = W(\Phi)$$

where W is an ordinary function. Explain why this field equation is Lorentz invariant and invariant under supersymmetry.

Assume now that $W(\Phi) = \mathrm{e}^{\Phi}$, and that Φ has the expansion in component fields

$$\Phi = \phi \ + \ i\theta_-\psi_+ \ + \ i\theta_+\psi_- \ + \ i\theta_-\theta_+F\,.$$

Find the field equations satisfied by the component fields ϕ, ψ_+ and ψ_- after the auxiliary field F is eliminated.

4 In the context of supersymmetric field theory in 3+1 dimensions, write brief notes on

a) 2-component Weyl spinors,

b) ${\sigma^\mu}_{\alpha\beta}$ and the supersymmetry algebra,

c) Dirac and Majorana mass terms,

d) the chiral constraint $\overline{D}_\alpha \Phi = 0$,

e) the consistency of the equation $\overline{D}\,\overline{D}\,\Phi^* = 4m\Phi + 4\lambda\Phi^2$ for an L-superfield.

[You need not find or discuss the equations for the component fields.]

$$\left[\textit{Note:} \quad \overline{D}_\alpha = \frac{\partial}{\partial \overline{\theta}^\alpha} + i\theta^\beta {\sigma^\mu}_{\beta\alpha}\partial_\mu \,. \right]$$

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