UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2009 1:30 pm to 4:30 pm

PAPER 4

CHARACTER THEORY OF FINITE GROUPS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

Results from lectures and exercise sheets may be used without proof — except in cases where you are explicitly asked to prove them — provided their use is properly indicated. In some cases you are asked to give full statements of such results, and you should do so.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let G be a finite group. Suppose that H is a Frobenius complement in G. Let

$$N = \left(G \setminus \bigcup_{g \in G} H^g \right) \cup \{1\}.$$

(i) Suppose θ is a class function on H and $\theta(1) = 0$. Show that $(\theta^G)_H = \theta$.

(ii) Prove that N is a normal subgroup of G and that $NH = G$. (*You may assume that* $|N| = |G : H|$.)

(iii) Let χ be an irreducible character of G. Suppose $N \subseteq \ker \chi$ and $\langle \chi_H, 1_H \rangle > 0$. Prove that $\chi = 1_G$.

(iv) Let $h \in H \setminus \{1\}$ and $x \in N$. Prove that there exists $y \in N$ such that $[h, y] = x$, where $[h, y] = h^{-1}y^{-1}hy$. (*Hint. An injective map of a finite set to itself is surjective.*)

(v) Suppose N is abelian. Using (iv) or otherwise, show that, for every non-trivial $\phi \in \mathrm{Irr}(N)$, the inertia group $I_G(\phi)$ is equal to N.

2 Let N be a normal subgroup of a finite group G. Suppose $\theta \in \mathrm{Irr}(N)$, and let $T = I_G(\theta)$ be the inertia group. Prove that

(a) if $\xi \in \mathrm{Irr}(T|\theta)$ then $\xi^G \in \mathrm{Irr}(G|\theta)$;

(b) the map $\xi \mapsto \xi^G$ is a bijection of $\mathrm{Irr}(T|\theta)$ onto $\mathrm{Irr}(G|\theta)$.

A group K is called *metabelian* if it has an abelian normal subgroup L such that K/L is abelian. Prove that every metabelian finite group is an M-group.

3 Throughout the question, λ and μ denote partitions of a fixed integer n. Explain what it means to say that λ dominates μ (that is, $\lambda \trianglerighteq \mu$).

Let X_λ be the set of all λ-tabloids and consider the corresponding permutation $\mathbb{C}S_n$-module $\mathbb{C}X_\lambda$. If t is a λ-tableau, let

$$\kappa_t = \sum_{g \in C_t} \operatorname{sgn}(g) g,$$

where C_t is the column stabiliser of t. Set $e_t = \kappa_t[t] \in M^\lambda$.

Explain briefly why $ge_t = e_{gt}$ for all $g \in S_n$. Give a definition of the Specht module S^λ.

In what follows, you may use results on the value of $\kappa_t[s]$ — where t is a λ-tableau and s is a μ-tableau — without proof, provided you state them clearly. You may also assume that the inner product on M^λ given by

$$\langle [t],[s] \rangle = \begin{cases} 1 & \text{if } [t]=[s], \\ 0 & \text{otherwise} \end{cases}$$

satisfies $\langle \kappa_t u, v\rangle = \langle u, \kappa_t v\rangle$ whenever $u, v \in M^\lambda$ and t is a λ-tableau. The orthogonal complement below is taken with respect to this inner product.

(i) Let U be a submodule of M^λ. Prove that either $U \supseteq S^\lambda$ or $U \subseteq (S^\lambda)^\perp$. Deduce that S^λ is simple.

(ii) Show that if $\operatorname{Hom}_{\mathbb{C}S_n}(S^\lambda, M^\mu) \neq 0$ then $\lambda \trianglerighteq \mu$. Also show that $\dim \operatorname{Hom}_{\mathbb{C}S_n}(S^\lambda, M^\lambda) = 1$.

(iii) For all λ and μ, denote by χ^λ the character of S_n afforded by S^λ, and by ξ^μ the character afforded by M^μ. Prove that

$$\xi^\mu = \sum_{\lambda \trianglerighteq \mu} \langle \xi^\mu, \chi^\lambda \rangle \chi^\lambda$$

and that $\langle \xi^\mu, \chi^\mu \rangle = 1$.

(iv) Hence, or otherwise, prove that $\chi^\lambda(g) \in \mathbb{Z}$ for all $g \in S_n$ and all partitions λ of n.

(v) Suppose $n \equiv 3 \mod 4$. Show that there exist an irreducible character θ of the alternating group A_n and an element $g \in A_n$ such that $\theta(g) \notin \mathbb{R}$.

4 Give definitions of a p-elementary group and an elementary group. State Brauer's characterisation of characters.

Let N be a normal subgroup of a finite group G. Suppose $\theta \in \mathrm{Irr}(N)$ is G-invariant and $\theta(1)$ is coprime to $|G : N|$. Assume that $\det \theta$ can be extended to a character μ of G. Prove that there exists a generalised character χ of G such that $\chi_N = \theta$. (*You may use the following result: if, in addition to the hypotheses above, G/N is solvable, then there exists a unique $\chi \in \mathrm{Irr}(G)$ such that $\chi_N = \theta$ and $\det \chi = \mu$.*)

Let π be a set of prime numbers, and let π' be the set of those primes that do not lie in π. Show that every elementary group is a direct product of a π-group and a π'-group.

Let K be a finite group. Consider the sets

$$\begin{aligned} A &= \{g \in K : g \text{ is a } \pi\text{-element and } g \neq 1\} \qquad \text{and} \\ B &= \{g \in K : g \text{ is a } \pi'\text{-element and } g \neq 1\}. \end{aligned}$$

Assume $K = A \cup B \cup \{1\}$. Prove that there exists a generalised character ξ of K such that $\xi(a) = 1$ for all $a \in A$ and $\xi(b) = 0$ for all $b \in B$.

END OF PAPER