UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2009 9:00 am to 11:00 am

PAPER 39

OPTIMAL INVESTMENT

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Baron Kiwi, the shoe-polish billionaire, appoints an investment manager to handle his investments for him. His initial wealth is w_0, and the manager can invest in a riskless asset returning a constant interest rate r, and in d risky assets, whose prices $S_t \equiv (S_t^1, \ldots, S_t^d)^T$ evolve as log-Brownian motions:

$$dS_t^i = S_t^i(\sum_{j=1}^{d} \sigma_{ij} dW_t^j + \mu_i dt) \qquad (i = 1, \ldots, d),$$

where the W^j are independent standard Brownian motions. The wealth process generated by the manager satisfies

$$dw_t = rw_t dt + \sum_i w_t \pi_t^i (\sum_j \sigma_{ij} dW_t^j + (\mu_i - r)dt),$$

where π_t^i denotes the fraction of wealth invested in asset i at time t. Baron Kiwi wants the investment manager to increase his wealth by time T, when his daughter is due to buy a University. If the manager's (admissible) investment results in a wealth w_T at time T, then the Baron has promised to pay the manager

$$y_T \equiv aw_T \exp(-\tfrac{1}{2}\varepsilon \int_0^T |\sigma^T \pi_s|^2 \, ds),$$

where $a > 0$, $\varepsilon > 0$, and the Baron's intention in reducing the payment by the exponential factor is to make the manager less keen to follow risky strategies.

If the manager's objective is to maximise $EU(y_T)$, where

$$U(x) = \tfrac{x^{1-R}}{1-R}$$

for some positive $R \neq 1$, prove that his optimal policy is exactly that which would be followed by an agent with utility

$$U_0(x) = \tfrac{x^{1-R-\varepsilon}}{1-R-\varepsilon}$$

and objective to maximize $EU_0(w_T)$.

Determine this optimal policy explicitly.

2 An agent may invest in a single risky asset whose price process S evolves as

$$dS_t = S_t(\sigma dW_t + \mu dt)$$

and in a riskless bank account. The interest paid on the bank account is at rate $r + \varepsilon > r$ until a random time τ which has an exponential distribution with mean λ^{-1}, and after that time the interest rate falls to r and stays there. The agent's objective is to maximise

$$E \int_0^\infty e^{-\rho t} U(c_t)\, dt,$$

where c_t is the rate of consumption withdrawal, $\rho > 0$ is constant, and $U(x) = x^{1-R}/(1-R)$ for some $R > 0$ different from 1.

(i) Consider first the case where the interest rate has already changed. Write down the agent's value function and optimal investment/consumption policy. [*You may use without derivation any standard results from the course.*]

(ii) Now consider the case where the interest rate has not yet changed. Obtain an equation which characterises the agent's value function and optimal investment/consumption policy, and find the value function as explicitly as you can.

3 (i) Suppose that an agent may invest in a riskless bank account yielding interest at constant rate r, and in a stock, whose price S_t at time t evolves as

$$dS_t = S_t(\sigma dW_t + \mu dt).$$

If the agent aims to maximise the objective

$$(*) \qquad E \int_0^\infty e^{-\rho t} U(c_t)\, dt,$$

where $U'(x) = x^{-R}$ for some positive $R \neq 1$, and c_t is rate of consumption withdrawal, write down the dynamics for the agent's wealth, and by finding the Hamilton-Jacobi-Bellman equations for this problem, or otherwise, show that the agent's optimal policy is to invest a fixed proportion π_M of his wealth in the risky asset, and to consume at a rate $c_t = \gamma_M w_t$ for some constant γ_M, where you should identify π_M and γ_M explicitly in terms of the parameters of the problem.

[*You should assume that* $\rho + (R-1)(r + (\mu - r)^2/2\sigma^2 R) > 0$. *You are not required to give a verification proof.*]

(ii) Suppose that there is a single productive asset in the world, which generates consumption goods at rate δ_t, where for some constants $\sigma > 0$, α,

$$d\delta_t = \delta_t(\sigma dW_t + \alpha dt).$$

There is a market in shares in the productive asset, and in riskless borrowing/lending of the consumption good.

Consider a representative-agent economy, where the single agent has objective (*). Show that if the share price process $(S_t)_{t\geqslant 0}$ is

$$S_t = \frac{\delta_t}{\rho + (R-1)(\alpha - \sigma^2 R/2)}$$

and the riskless rate of interest is constant, equal to

$$r^* = \rho + \alpha R - \tfrac{1}{2}(1+R)\sigma^2 R$$

then the markets clear; the representative agent faced with these prices will at all times hold one share in the productive asset, and will put no wealth into riskless borrowing/lending. Show also that the optimal consumption process is δ_t.

[*You should assume that* $\rho + (R-1)(\alpha - \sigma^2 R/2) > 0$.]

4 (i) Suppose that $X_t = W_t + \alpha t$, where W is a standard Brownian motion, and let $\mathcal{X}_t \equiv \sigma(X_u : 0 \leqslant u \leqslant t)$. The drift α is constant but unknown; it takes one of the values a or $-a$, where $a > 0$. Each of the two possible values has equal prior probability. Mr Bayes observes the process X, and makes inference on the unknown drift α. Show that after observing until time t his posterior for α takes the form

$$P[\, \alpha = a \mid \mathcal{X}_t \,] = \frac{e^{aX_t}}{e^{aX_t} + e^{-aX_t}}.$$

Deduce that the dynamics of X in the filtration $(\mathcal{X}_t)$ can be expressed as

$$dX_t = d\hat{W}_t + a \tanh aX_t \, dt,$$

where $\hat{W}$ is a $(\mathcal{X}_t)$-Brownian motion.

(ii) Mr Bayes wishes to invest in a market where there is a riskless bank account yielding interest at constant rate r, and a single risky asset whose price S_t at time t is given as

$$S_t = S_0 \exp(\sigma X_t + (\mu - \tfrac{1}{2}\sigma^2)t),$$

where X is as in part (i). His objective is to maximize $E \int_0^\infty e^{-\rho t} U(c_t)\, dt$, where $\rho > 0$ is constant, c_t is his rate of consumption withdrawal, and $U'(x) = x^{-R}$ for some positive $R \neq 1$.

Derive the Hamilton-Jacobi-Bellman equation for Mr Bayes' value function, and obtain an expression for the optimal investment in the risky asset.

Assume now that $\mu > r$, and $0 < R < 1$. Another agent, Mr Smart, knows that in fact $\alpha = a$; his objective is the same as Mr Bayes'. By considering Mr Smart's value function V, and the process

$$Y_t = \int_0^t e^{-\rho t} U(c_t)\, dt + e^{-\rho t} V(w_t),$$

where c is Mr Bayes' optimal consumption process, and w is Mr Bayes' optimal wealth process, show that Y is a supermartingale.

Deduce that Mr Bayes' value cannot exceed Mr Smart's value.

END OF PAPER