UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2009 1:30 pm to 3:30 pm

PAPER 37

ACTUARIAL STATISTICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let Y be a positive random variable with all moments finite and with moment generating function $M_Y(t) = \mathbb{E}(e^{Yt})$. The cumulant generating function is defined to be $\kappa_Y(t) = \log_e(M_Y(t))$ and the jth cumulant of Y is $\kappa_{Y,j} = \kappa_Y^{(j)}(0)$. Show that $\kappa_{Y,1}$ and $\kappa_{Y,2}$ are the mean and variance of Y, and that $\kappa_{Y,3}$ is the skewness of Y, given by $\mathbb{E}\big((Y - \mathbb{E}(Y))^3\big)$.

Let N be the number of claims arriving at an insurance company in a year. The claims $X_1, X_2, \ldots$ are independent identically distributed positive random variables, independent of N. Let S be the total amount claimed during the year. Show that the cumulant generating function of S is $\kappa_S(t) = \kappa_N(\kappa_{X_1}(t))$, where κ_N and κ_{X_1} are the cumulant generating functions of N and X_1 respectively.

For each of the following distributions for N, find the cumulant generating function and skewness of S, in each case expressing the skewness explicitly in terms of the moments of X_1 and the parameters of the distribution of N.

(a) N has a Poisson distribution with mean λ;

(b) N has a geometric distribution where $\mathbb{P}(N = n) = (1-p)^n p$, $n = 0, 1, 2, \ldots$ $(0 < p < 1)$;

(c) N has a binomial distribution with mean np and variance $np(1-p)$ $(0 < p < 1)$.

In cases (a) and (b), show that S must have positive skewness. In case (c), by considering claims that are equal to a positive constant with probability one, give an example where S has negative skewness.

2 Let N be the number of claims arriving at a direct insurer in one accounting period. The sizes of the claims, $X_1, X_2, \ldots$, are independent identically distributed (iid) positive random variables, independent of N, with density f_{X_1} and distribution function F_{X_1}. The direct insurer takes out an excess of loss reinsurance contract with fixed retention level $M > 0$. Assume that $0 < \mathbb{P}(X_1 > M) < 1$. Write down the amounts Y_i and Z_i paid out by the direct insurer and the reinsurer respectively on a single claim X_i.

Let N_R be the number of non-zero Z_i's in one accounting period. By writing N_R as $\sum_{i=1}^{N} I_i$ for some iid random variables $I_1, I_2, \ldots$, show that the probability generating function of N_R is given by $G_{N_R}(z) = G_N(\alpha z + 1 - \alpha)$, where $G_N(z) = \mathbb{E}(z^N)$ and α is some number between 0 and 1 which you should specify in terms of M and F_{X_1}.

The total amount S_R paid out by the reinsurer in one accounting period can be written $S_R = \sum_{j=1}^{N_R} W_j$, where the W_j's are iid positive random variables independent of N_R. Find the distribution function of W_1 in terms of M and F_{X_1}. Write down the density of W_1.

Now assume that $\mathbb{P}(N = k) = (1-p)^k p$, $k = 0, 1, 2, \ldots$, for $0 < p < 1$, and that X_1 is exponentially distributed with mean $\mu > 0$. Find the distributions of W_1 and N_R. Hence identify the distribution of S_R, and, for $s > 0$, find the probability that the reinsurer's total payment in one accounting period exceeds s.

3 In a classical risk model, claims arrive in a Poisson process with rate $\lambda > 0$, the claim sizes have density $f(x)$ and mean μ, and the premium income rate is $c = (1+\theta)\lambda\mu$, where $\theta > 0$. Define the surplus process $U(t)$ and the probability of ruin $\psi(u)$ with initial capital $u \geqslant 0$.

Let $\phi(u) = 1 - \psi(u)$. Show that

$$\phi'(u) = \frac{\phi(u)}{(1+\theta)\mu} - \frac{1}{(1+\theta)\mu} \int_0^u \phi(u-x) f(x)\,dx.$$

Now suppose that $f(x) = 4xe^{-2x}$, $x > 0$, and $\theta = 2$. Show that

$$3\phi'(u) = \phi(u) - 4e^{-2u} I(u),$$

where $I(u) = \int_0^u (u-t)\phi(t)e^{2t}\,dt$.

Show that

$$3\phi''(u) = -5\phi'(u) + 2\phi(u) - 4e^{-2u} \int_0^u \phi(t)e^{2t}\,dt,$$

and hence show that

$$3\phi'''(u) + 11\phi''(u) + 8\phi'(u) = 0.$$

Given that $\phi(0) = \theta/(1+\theta)$ and that $\phi(u) \to 1$ as $u \to \infty$, find $\phi(u)$ and hence find $\psi(u)$.

4 A fleet of company cars is insured each year, and in year j the fleet has m_j cars (m_j known). Let X_j be the amount claimed per car in year j, $j = 1, \ldots, n$, where, conditional on a risk parameter θ, the X_j's are independent with common mean $\mathbb{E}(X_1 \mid \theta) = \mu(\theta)$, and variances $\text{var}(X_j \mid \theta) = \sigma^2(\theta)/m_j$. The credibility premium per car for year $n+1$ is defined to be $a_0 + \sum_{j=1}^n a_j X_j$ where the a_j's are chosen to minimise

$$\mathbb{E}\Big[\big(\mu(\theta) - a_0 - \sum_{j=1}^{n} a_j X_j\big)^2\Big].$$

Find the credibility premium per car in year $n+1$. Show that it is of the form

$$Z \frac{\sum_{j=1}^n m_j X_j}{\sum_{j=1}^n m_j} + (1 - Z)\mathbb{E}\left[\mu(\theta)\right],$$

and give an expression for Z.

Suppose that in an approximate model, the conditional distribution of X_j given θ is normal with mean θ and variance v/m_j, and suppose also that θ is normally distributed with mean μ and variance a. In this approximate model, find the credibility premium per car for year $n+1$.

Derive the Bayesian estimate of $\mu(\theta)$ under quadratic loss in the approximate model, and compare it with the credibility estimate above.

END OF PAPER