UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2009 9:00 am to 12:00 pm

## PAPER 32

## ADVANCED FINANCIAL MODELS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Consider a one-period $(d+1)$-asset market model where asset 0 is a numéraire asset. What is an arbitrage strategy? Show that there is no arbitrage if there exists a positive random variable $\rho$ such that

$$S_0^{(i)} = \mathbb{E}[\rho S_1^{(i)}] \tag{*}$$

for each $i \in \{0, \ldots, d\}$ where $S_t^{(i)}$ is the price at time $t$ of asset $i$.

What does it mean to say a contingent claim is attainable? Prove that if every contingent claim in this market is attainable, then there is at most one positive random variable $\rho$ satisfying equation (*).

Now consider a two-asset model with prices given by

$$\begin{tikzcd}
(S^{(0)}, S^{(1)}) & & (3,6) \\
& (4,5) \arrow[ur, "1/3"] \arrow[r, "1/3"'] \arrow[dr, "1/3"'] & (5,4) \\
& & (6,3)
\end{tikzcd}$$

Find all positive random variables $\rho$ satisfying equation (*). Show by example that there exists a claim that is not attainable.

**2**

Let $Z_1, Z_2, \ldots$ be a sequence of independent positive random variables. Suppose $Y_0$ is a positive constant, and let $Y_t = Z_1 Z_2 \cdots Z_t Y_0$ for $t \in \{1, 2, \ldots\}$. Fix $T > 0$ and let $U$ be the Snell envelope of the process $(Y_t)_{t \in \{0,\ldots,T\}}$. Show that there is a sequence of positive constants $c_0, \ldots, c_T$ such that

$$U_t = Y_t c_t.$$

Now consider a discrete-time, two-asset market model with bank account $B_t = (1+r)^t$, stock price $S_t = (1+R_1)(1+R_2)\cdots(1+R_t)S_0$, where $S_0$ is a positive constant and $R_1, R_2, \ldots$ is a sequence of independent random variables with identical distribution

$$\mathbb{P}(R_t = \varepsilon) = \frac{1}{2} + \frac{r}{2\varepsilon} \text{ and } \mathbb{P}(R_t = -\varepsilon) = \frac{1}{2} - \frac{r}{2\varepsilon}.$$

Here $r$ and $\varepsilon$ are constants such that $0 \leqslant r < \varepsilon < 1$. Show that there is no arbitrage in this market. You may use a standard no-arbitrage theorem without proof as long as it is carefully stated.

In this market, there is an American contingent claim with maturity $T > 0$, which pays $\xi_t = S_t^2$ if exercised at time $t$, for any $0 \leqslant t \leqslant T$. Using the fact that the market is complete and a standard theorem on American options, find the time 0 replication cost of this option in terms of $S_0$, $T$, $r$ and $\varepsilon$. How many shares of the stock should the seller of the option hold between time 0 and time 1 to hedge the optimally exercised claim?

**3** Suppose a market has $d+1$ assets with prices given by

$$dB_t = r_t B_t dt$$

and

$$dS_t^{(i)} = S_t^{(i)} \left( \mu_t^{(i)}\, dt + \sum_{j=1}^{d} \sigma_t^{(i,j)} dW_t^{(j)} \right)$$

for $i \in \{1, \ldots, d\}$ where the adapted processes $(r_t)_{t \in \mathbb{R}_+}$, $(\mu_t^{(i)})_{t \in \mathbb{R}_+}$ and $(\sigma_t^{(i,j)})_{t \in \mathbb{R}_+}$ are bounded and continuous, and the Brownian motions $(W_t^{(i)})_{t \in \mathbb{R}_+}$ are independent.

What is an admissible trading strategy? What is an arbitrage? Show that if the $d \times d$ matrix-valued process $(\sigma_t^{-1})_{t \in \mathbb{R}_+}$ is bounded, then the market has no arbitrage. Standard results from stochastic calculus may be used without proof, but they must be stated clearly.

A market is said to satisfy the Law of One Price if it has the property that $S_T^{(i)} = S_T^{(j)}$ almost surely implies $S_t^{(i)} = S_t^{(j)}$ almost surely for all $0 \leqslant t \leqslant T$. Give an example of a continuous time market model with no arbitrage which does *not* obey the Law of One Price. You may use without proof the following fact about a Brownian motion $X$: for every $k \in \mathbb{R}$, the hitting time $\tau = \inf\{t \geqslant 0 : X_t = k\}$ is finite almost surely.

**4** Consider a two-asset model where asset 0 is cash, so that the price of asset 0 is $B_t = 1$ for all $t \geqslant 0$. Asset 1 has prices given by

$$dS_t = a(S_t)dW_t$$

where the given function $a$ is positive and smooth, and such that $a$ and its derivative $a'$ are bounded. Let $\xi_t$ be the time-$t$ price of a (European) call option with maturity $T$ and strike $K$. Finally, let $V : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ satisfy the partial differential equation

$$\frac{\partial}{\partial t}V(t,S) + \frac{a(S)^2}{2}\frac{\partial^2}{\partial S^2}V(t,S) = 0$$

with boundary condition

$$V(T,S) = (S-K)^+.$$

Show that there is no arbitrage in the augmented market if $\xi_t = V(t,S_t)$. A standard no-arbitrage theorem can be used without proof as long as it is carefully stated.

Show that the call option can be replicated by holding $\pi_t = U(t,S_t)$ units of stock, where $U : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{aligned}&\frac{\partial}{\partial t}U(t,S) + a(S)a'(S)\frac{\partial}{\partial S}U(t,S) + \frac{a(S)^2}{2}\frac{\partial^2}{\partial S^2}U(t,S) = 0\\ &U(T,S) = \mathbb{1}_{\{S \geqslant K\}}.\end{aligned}$$

You may assume that $U$ and $V$ are smooth in $[0,T) \times \mathbb{R}$.

Let $(Z_t)_{t \geqslant 0}$ be the martingale defined by $Z_0 = 1$ and

$$dZ_t = Z_t a'(S_t)dW_t.$$

Let $M_t = Z_t \pi_t$. Show that $M$ is a local martingale. Assuming $M$ is a true martingale, derive the inequality $0 \leqslant \pi_t \leqslant 1$ almost surely.

**5**

Let $Z \sim N(0,1)$ be a standard normal random variable, and let

$$F(v,m) \;=\; \mathbb{E}[(e^{-v/2+\sqrt{v}Z} - m)^+]\,.$$

Express $F(v,m)$ in terms of the standard normal distribution function. Hence, or otherwise, prove the identity

$$F(v,m) \;=\; 1 - m + m\;F(v,1/m)\,.$$

Now consider a two asset model, where asset 0 is a bank account $B_t = e^{rt}$ for a positive constant $r$, and asset 1 is a stock with prices $S_t$ given by

$$S_t \;=\; S_0\, e^{(r-\sigma^2/2)\,t+\sigma W_t}$$

for a positive constant $\sigma$ and Brownian motion $W$. Show that there is no arbitrage if the time-$t$ price of a call with maturity $T$ and strike $K$ is given by

$$C_t(T,K) \;=\; S_t\, F[(T-t)\sigma^2, Ke^{-r(T-t)}/S_t]\,.$$

You may use a standard no-arbitrage theorem without proof as long as it is carefully stated.

Now, assuming that $C_t(T,K)$ is as above, show that there is no arbitrage if the time $t$ price of a put option with maturity $T$ and strike $K$ is given by the put-call parity formula

$$P_t(T,K) \;=\; Ke^{-r(T-t)} - S_t + C_t(T,K)\,.$$

Hence, establish the put-call symmetry formula

$$P_t(T,K) \;=\; Ke^{-\gamma\,(T-t)}\, F[(T-t)\,\sigma^2,\; S_t\, e^{-r\,(T-t)}/K]\,.$$

**6** What is a (zero-coupon) bond? How are the bond prices related to the forward rates?

Consider a short interest rate process $(r_t)_{t\in\mathbb{R}_+}$ satisfying the following stochastic differential equation:

$$dr_t = a(r_t)dt + b(r_t)dW_t$$

for two given smooth functions $a$ and $b$ and a Brownian motion $W$. Let the function $F$ satisfy the following integral-differential equation

$$\frac{\partial F}{\partial \theta}(\theta, r) = a(r)\frac{\partial F}{\partial r}(\theta, r) + \frac{b(r)^2}{2}\frac{\partial^2 F}{\partial r^2}(\theta, r) - b(r)^2\frac{\partial F}{\partial r}(\theta, r)\int_0^\theta \frac{\partial F}{\partial r}(s, r)ds$$

with initial condition $F(0, r) = r$. Show that there is no arbitrage if the forward rates are given by $f_t(T) = F(T - t, r_t)$. You may use a standard no-arbitrage theorem without proof as long as it is carefully stated.

Now suppose $a(r) = a_0$ and $b(r) = b_0$ for some constants $a_0$ and $b_0$. Show that there is no arbitrage if $f_t(T) = A(T-t)r_t + B(T-t)$ for some functions $A$ and $B$, which you should find.

## END OF PAPER