UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2009 9:00 am to 12:00 pm

PAPER 32

ADVANCED FINANCIAL MODELS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a one-period $(d+1)$-asset market model where asset 0 is a numéraire asset. What is an arbitrage strategy? Show that there is no arbitrage if there exists a positive random variable ρ such that

$$S_0^{(i)} = \mathbb{E}[\rho S_1^{(i)}] \tag{*}$$

for each $i \in \{0, \ldots, d\}$ where $S_t^{(i)}$ is the price at time t of asset i.

What does it mean to say a contingent claim is attainable? Prove that if every contingent claim in this market is attainable, then there is at most one positive random variable ρ satisfying equation (*).

Now consider a two-asset model with prices given by

$$(S^{(0)}, S^{(1)})$$

$$\begin{tikzcd} & (3,6) \\ (4,5) \arrow[ur, "1/3"] \arrow[r, "1/3"'] \arrow[dr, "1/3"'] & (5,4) \\ & (6,3) \end{tikzcd}$$

Find all positive random variables ρ satisfying equation (*). Show by example that there exists a claim that is not attainable.

2

Let $Z_1, Z_2, \ldots$ be a sequence of independent positive random variables. Suppose Y_0 is a positive constant, and let $Y_t = Z_1 Z_2 \cdots Z_t Y_0$ for $t \in \{1, 2, \ldots\}$. Fix $T > 0$ and let U be the Snell envelope of the process $(Y_t)_{t \in \{0,\ldots,T\}}$. Show that there is a sequence of positive constants $c_0, \ldots, c_T$ such that

$$U_t = Y_t c_t.$$

Now consider a discrete-time, two-asset market model with bank account $B_t = (1+r)^t$, stock price $S_t = (1+R_1)(1+R_2)\cdots(1+R_t)S_0$, where S_0 is a positive constant and $R_1, R_2, \ldots$ is a sequence of independent random variables with identical distribution

$$\mathbb{P}(R_t = \varepsilon) = \frac{1}{2} + \frac{r}{2\varepsilon} \text{ and } \mathbb{P}(R_t = -\varepsilon) = \frac{1}{2} - \frac{r}{2\varepsilon}.$$

Here r and ε are constants such that $0 \leqslant r < \varepsilon < 1$. Show that there is no arbitrage in this market. You may use a standard no-arbitrage theorem without proof as long as it is carefully stated.

In this market, there is an American contingent claim with maturity $T > 0$, which pays $\xi_t = S_t^2$ if exercised at time t, for any $0 \leqslant t \leqslant T$. Using the fact that the market is complete and a standard theorem on American options, find the time 0 replication cost of this option in terms of S_0, T, r and ε. How many shares of the stock should the seller of the option hold between time 0 and time 1 to hedge the optimally exercised claim?

3 Suppose a market has $d+1$ assets with prices given by

$$dB_t = r_t B_t dt$$

and

$$dS_t^{(i)} = S_t^{(i)} \left(\mu_t^{(i)}\, dt + \sum_{j=1}^{d} \sigma_t^{(i,j)} dW_t^{(j)} \right)$$

for $i \in \{1, \ldots, d\}$ where the adapted processes $(r_t)_{t \in \mathbb{R}_+}$, $(\mu_t^{(i)})_{t \in \mathbb{R}_+}$ and $(\sigma_t^{(i,j)})_{t \in \mathbb{R}_+}$ are bounded and continuous, and the Brownian motions $(W_t^{(i)})_{t \in \mathbb{R}_+}$ are independent.

What is an admissible trading strategy? What is an arbitrage? Show that if the $d \times d$ matrix-valued process $(\sigma_t^{-1})_{t \in \mathbb{R}_+}$ is bounded, then the market has no arbitrage. Standard results from stochastic calculus may be used without proof, but they must be stated clearly.

A market is said to satisfy the Law of One Price if it has the property that $S_T^{(i)} = S_T^{(j)}$ almost surely implies $S_t^{(i)} = S_t^{(j)}$ almost surely for all $0 \leqslant t \leqslant T$. Give an example of a continuous time market model with no arbitrage which does *not* obey the Law of One Price. You may use without proof the following fact about a Brownian motion X: for every $k \in \mathbb{R}$, the hitting time $\tau = \inf\{t \geqslant 0 : X_t = k\}$ is finite almost surely.

4 Consider a two-asset model where asset 0 is cash, so that the price of asset 0 is $B_t = 1$ for all $t \geqslant 0$. Asset 1 has prices given by

$$dS_t = a(S_t)dW_t$$

where the given function a is positive and smooth, and such that a and its derivative a' are bounded. Let ξ_t be the time-t price of a (European) call option with maturity T and strike K. Finally, let $V : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ satisfy the partial differential equation

$$\frac{\partial}{\partial t}V(t,S) + \frac{a(S)^2}{2}\frac{\partial^2}{\partial S^2}V(t,S) = 0$$

with boundary condition

$$V(T,S) = (S-K)^+.$$

Show that there is no arbitrage in the augmented market if $\xi_t = V(t,S_t)$. A standard no-arbitrage theorem can be used without proof as long as it is carefully stated.

Show that the call option can be replicated by holding $\pi_t = U(t,S_t)$ units of stock, where $U : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{aligned}&\frac{\partial}{\partial t}U(t,S) + a(S)a'(S)\frac{\partial}{\partial S}U(t,S) + \frac{a(S)^2}{2}\frac{\partial^2}{\partial S^2}U(t,S) = 0\\ &U(T,S) = \mathbb{1}_{\{S \geqslant K\}}.\end{aligned}$$

You may assume that U and V are smooth in $[0,T) \times \mathbb{R}$.

Let $(Z_t)_{t\geqslant 0}$ be the martingale defined by $Z_0 = 1$ and

$$dZ_t = Z_t a'(S_t)dW_t.$$

Let $M_t = Z_t \pi_t$. Show that M is a local martingale. Assuming M is a true martingale, derive the inequality $0 \leqslant \pi_t \leqslant 1$ almost surely.

5

Let $Z \sim N(0,1)$ be a standard normal random variable, and let

$$F(v,m) = \mathbb{E}[(e^{-v/2+\sqrt{v}Z} - m)^+] .$$

Express $F(v,m)$ in terms of the standard normal distribution function. Hence, or otherwise, prove the identity

$$F(v,m) = 1 - m + m\, F(v, 1/m) .$$

Now consider a two asset model, where asset 0 is a bank account $B_t = e^{rt}$ for a positive constant r, and asset 1 is a stock with prices S_t given by

$$S_t = S_0\, e^{(r - \sigma^2/2)\,t + \sigma W_t}$$

for a positive constant σ and Brownian motion W. Show that there is no arbitrage if the time-t price of a call with maturity T and strike K is given by

$$C_t(T,K) = S_t\, F[(T-t)\sigma^2, Ke^{-r(T-t)}/S_t] .$$

You may use a standard no-arbitrage theorem without proof as long as it is carefully stated.

Now, assuming that $C_t(T,K)$ is as above, show that there is no arbitrage if the time t price of a put option with maturity T and strike K is given by the put-call parity formula

$$P_t(T,K) = Ke^{-r(T-t)} - S_t + C_t(T,K) .$$

Hence, establish the put-call symmetry formula

$$P_t(T,K) = Ke^{-\gamma\,(T-t)}\, F[(T-t)\,\sigma^2,\, S_t\, e^{-r\,(T-t)}/K] .$$

6 What is a (zero-coupon) bond? How are the bond prices related to the forward rates?

Consider a short interest rate process $(r_t)_{t\in\mathbb{R}_+}$ satisfying the following stochastic differential equation:

$$dr_t = a(r_t)dt + b(r_t)dW_t$$

for two given smooth functions a and b and a Brownian motion W. Let the function F satisfy the following integral-differential equation

$$\frac{\partial F}{\partial \theta}(\theta, r) = a(r)\frac{\partial F}{\partial r}(\theta, r) + \frac{b(r)^2}{2}\frac{\partial^2 F}{\partial r^2}(\theta, r) - b(r)^2\frac{\partial F}{\partial r}(\theta, r)\int_0^\theta \frac{\partial F}{\partial r}(s, r)ds$$

with initial condition $F(0, r) = r$. Show that there is no arbitrage if the forward rates are given by $f_t(T) = F(T-t, r_t)$. You may use a standard no-arbitrage theorem without proof as long as it is carefully stated.

Now suppose $a(r) = a_0$ and $b(r) = b_0$ for some constants a_0 and b_0. Show that there is no arbitrage if $f_t(T) = A(T-t)r_t + B(T-t)$ for some functions A and B, which you should find.

END OF PAPER