UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2009 1:30 pm to 4:30 pm

PAPER 31

ADVANCED PROBABILITY

Attempt no more than ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State Doob's upcrossing inequality for a martingale $(X_n : n = 1, 2, 3, \ldots)$.

Deduce that, if (X_n) is bounded in L^1, then (X_n) converges almost surely to some $X_\infty \in L^1$. What extra condition is needed for L^1-convergence?

Give an example to show that this extra condition is not redundant.

2 What is a Lévy measure? Assume K is a Lévy measure with the property that

$$\int_{[-1,1]} |y| \, K(dy) < \infty.$$

State the Lévy-Khinchin formula in a form that involves the integral

$$\int \left(e^{iu\,y} - 1\right) K(dy) \, .$$

Assume a Lévy-process $(X_t : t \geqslant 0)$ has characteristic function at time 1 given by

$$\mathbb{E}\left(\exp\left(iuX_1\right)\right) = \frac{1}{1 + u^2/2}.$$

Why does this determine the law of the entire process? Determine the corresponding Lévy-measure.

[*Hint: look for K with density $f(|y|) / |y|$ for $y \neq 0$; you may want to use the fact that the Fourier sine transform of $f(y) = e^{-y}$ is given by*

$$\int_0^\infty \sin(u\,y)\, f(y)\, dy = \frac{u}{1+u^2}.]$$

3 State Prohorov's theorem for a sequence of probability measures $(\mu_n : n = 1, 2, 3, \ldots)$ on the real line.

Let μ be a probability measure with characteristic function ϕ. Show that there exists C such that for all $\lambda > 0$,

$$\mu\left(\{y : |y| \geqslant \lambda\}\right) \ \leqslant\ C\lambda \int_0^{1/\lambda} \left(1 - \mathrm{Re}\phi\left(u\right)\right)\, du\,.$$

[*You may assume that, for all* $t \geqslant 1$,

$$\frac{1}{t}\int_0^t \left(1 - \cos v\right)\, dv \ \geqslant\ 1 - \sin 1\,.]$$

Now let $\mu, \mu_1, \mu_2, \ldots$ be a sequence of probability measures on the real line with characteristic functions $\phi, \phi_1, \phi_2, \ldots$, and assume $\phi_n\left(u\right) \to \phi\left(u\right)$ as $n \to \infty$ for all real u. Show that the sequence of measures (μ_n) is tight.

Hence show that μ_n converges weakly to μ.

Let $(X_n : n = 1, 2, 3, \ldots)$ be a sequence of independent and identically distributed random variables, with characteristic function ϕ. Set $S_n = X_1 + \ldots + X_n$ and assume $S_n/n \to a$ in probability. Show that $Q'\left(0\right)$ exists and determine its value. Discuss to what extent integrability of X_n is needed in your argument.

4 Let $B = (B_t : t \geqslant 0)$ be a Brownian motion started at zero and f a sufficiently nice function such that

(1) $$M_t^f = f(B_t) - f(0) - \int_0^t \frac{1}{2} f''(B_s)\, ds \text{ is a continuous martingale.}$$

Verify the formula

(2) $$\mathbb{E}\left[\left|M_t^f\right|^2\right] = \int_0^t \mathbb{E}\left(\left|f'(B_s)\right|^2\right)$$

for $f(x) = x$ and $f(x) = x^2$.

In the sequel you may assume that (1) and (2) hold true for functions $f_n(x)$ given by

$$f_n(x) = \begin{cases} |x| \text{ for } |x| \geqslant 1/n \\ \frac{n}{2}x^2 + \frac{1}{2n} \text{ else} \end{cases}$$

where $n \in \{1, 2, 3, \ldots\}$. Show that there exists a continuous martingale $M = (M_t : t \geqslant 0)$ such that

$$\mathbb{E}\left[\sup_{s \leqslant t} \left|M_s^{f_n} - M_s\right|^2\right] \to 0.$$

Conclude that, as $n \to \infty$,

$$\int_0^t \left(n 1_{|B_s| < 1/n}\right) ds \to |B_t| - M_t \text{ almost surely.}$$

[*Hint: use Doob's L^2-inequality*]

5 Write an essay explaining how both Markov chains and Brownian motion may be characterized in terms of certain martingales, and how martingales can be used to analyse further the behaviour of these processes. You must include a proof that 3-dimensional Brownian motion is transient.

5 Write an essay explaining how both Markov chains and Brownian motion may be characterized in terms of certain martingales, and how martingales can be used to analyse further the behaviour of these processes. You must include a proof that 3-dimensional Brownian motion is transient.

END OF PAPER