UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2009 1:30 pm to 4:30 pm

PAPER 30

STOCHASTIC NETWORKS

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Suppose that the stream of customers arriving at a queue forms a Poisson process of rate ν and that there are two servers who differ in efficiency. Specifically, suppose that a customer's service time at server i is exponentially distributed with parameter μ_i, for $i = 1, 2$, where $\mu_1 + \mu_2 > \nu$. The sequences of interarrival times at the queue, and of service times at each of the two servers, are independent. If a customer arrives to find both servers free she is equally likely to be allocated to either server.

Find the stationary probability that both servers are busy.

Prove that the stream of customers departing from the queue forms a Poisson process.

(ii) Define an *open migration process.* Establish the form of the equilibrium distribution, giving conditions for its existence.

In a stationary open migration process, each colony j acts as a single-server queue at which services time are exponentially distributed with parameter ϕ_j. Let α_j be the mean arrival rate at queue j. If $\phi_j, j = 1, 2, \ldots, J$, can be chosen subject to the constraint

$$\sum_{j=1}^{J} \phi_j = F,$$

find the choice, in terms of F and α_j, $j = 1, 2, \ldots, J$, that minimizes the mean number of individuals in the system.

2 Describe briefly a mathematical model for a loss network with fixed routing, and obtain an exact expression for the probability that an arriving call for route r is blocked.

A network consists of three nodes, with each pair of nodes connected by a link. A call in progress between two nodes may be routed on the direct link between the nodes, or on the two-link path through the third node. A call in progress can be rerouted if this will allow an additional arriving call to be accepted. Describing carefully the modelling assumptions you make, obtain an exact expression for the probability an arriving call is blocked, for each of three possible node pairs. Show that the network is equivalent to a certain loss network with fixed routing, to be identified.

3 Define a *Wardrop equilibrium* for the flows $x = (x_r, r \in R)$ in a congested network with routes $r \in R$ and links $j \in J$.

Show that if the delay $D_j(y_j)$ at link j is a continuously differentiable, strictly increasing function of the throughput, y_j, of the link j then a Wardrop equilibrium exists and solves an optimization problem of the form

$$\begin{array}{rl} \text{minimize} & \displaystyle\sum_{j \in J} \int_0^{y_j} D_j(u)du \\ \text{over} & x \geqslant 0, \quad y \\ \text{subject to} & Hx = f, \qquad Ax = y \ , \end{array}$$

where $f = (f_s, s \in S)$ and f_s is the (fixed) aggregate flow between source-sink pair s. What are the matrices A and H? Are the equilibrium throughputs, y_j, unique? Are the equilibrium flows, x_r, unique? Justify your answers.

Suppose now that, in addition to the delay $D_j(y_j)$, users of link j incur a traffic-dependent toll $T_j(y_j)$, and suppose each user selects a route in an attempt to minimize the sum of its toll and its delays. Show that it is possible to chose the functions $T_j(\cdot)$ so that the equilibrium flow pattern minimizes the average delay in the network.

4 The dynamical system

$$\frac{d}{dt}x_r(t) = \kappa_r \left(w_r - x_r(t) \sum_{j \in r} \mu_j(t) \right) \qquad r \in R$$

$$\mu_j(t) = p_j \left(\sum_{s : j \in s} x_s(t) \right) \qquad j \in J$$

is proposed as a model for a communication network, where R is a set of routes, J is a set of resources, $\kappa_r > 0$ and $w_r > 0$, $r \in R$, and $p_j(\cdot)$, $j \in J$, are non-negative, continuous, increasing functions.

Provide a brief interpretation of this model, in terms of feedback signals generated by resources and acted upon by users.

By considering the function

$$U(x) = \sum_{r \in R} w_r \log x_r - \sum_{j \in J} \int_0^{\sum_{s:j \in s} x_s} p_j(y) dy$$

or otherwise, show that all trajectories of the dynamical system converge towards a unique equilibrium point.

An alternative dynamical system

$$\frac{d}{dt}\mu_j(t) = \kappa_j \, \mu_j(t) \left(\sum_{r : j \in r} x_r(t) - q_j\left(\mu_j(t)\right) \right) \qquad j \in J$$

$$x_r(t) = \frac{w_r}{\sum_{k \in r} \mu_k(t)} \qquad r \in R$$

is proposed, where, for $j \in J$, $\kappa_j > 0$, and $q_j(\cdot)$ is a continuous, strictly increasing function with $q_j(0) = 0$. Show that this dynamical system has a unique equilibrium point.

Discuss briefly any possible relationship between the equilibrium points of the two dynamical systems.

5 Let J be a set of resources, and R a set of routes, where a route $r \in R$ identifies a subset of J. Let C_j be the capacity of resource j, and suppose the number of flows in progress on each route is given by the vector $n = (n_r, r \in R)$. Define a proportionally fair rate allocation.

Consider a *linear* network with resources $J = \{1, 2, \ldots, I\}$, each of unit capacity, and routes $R = \{0, 1, 2, \ldots, I\}$ where we use the symbol 0 to represent a route $\{1, 2, \ldots, I\}$ which traverses the entire set of resources, and the symbol i to represent a route $\{i\}$ through a single resource, for $i = 1, 2, \ldots, I$. Show that under a proportionally fair rate allocation

$$x_0 n_0 + x_i n_i = 1 \quad \text{if} \quad n_i > 0, \quad i = 1, 2, \ldots, I$$

and

$$x_0 = \frac{1}{n_0 + \sum_{i=1}^{I} n_i} \quad \text{if} \quad n_0 > 0.$$

Suppose now that flows describe the transfer of documents through the linear network above, that new flows originate as independent Poisson processes of rates $\nu_r, r \in R$, and that document sizes are independent and exponentially distributed with parameter μ_r for each route $r \in R$. Determine the transition intensities of the resulting Markov process $n = (n_r, r \in R)$. Show that the stationary distribution of the Markov process $n = (n_r, r \in R)$ takes the form

$$\pi(n) = (1 - \rho_0)^{1-I} \prod_{i=1}^{I} (1 - \rho_0 - \rho_i) \binom{\sum_{r=0}^{I} n_r}{n_0} \prod_{r=0}^{I} (\rho_r)^{n_r},$$

provided $\rho_0 + \rho_i < 1$, $i = 1, 2, \ldots, I$ where $\rho_r = \nu_r / \mu_r$. Show that under the distribution π,

$$\mathbb{E}(n_i) = \frac{\rho_i}{1 - \rho_0 - \rho_i} \qquad i = 1, 2, \ldots, I.$$

END OF PAPER