UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2009 1:30 pm to 4:30 pm

PAPER 3

COMMUTATIVE ALGEBRA

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the set Spec(A) for a ring A, and the *Zariski topology* on Spec(A). Show that the Zariski topology is in fact a topology.

State Zorn's Lemma, and use it to show that $\mathrm{Spec}(A) \neq \emptyset$ whenever $A \neq 0$.

Show that Spec(A) defines a contravariant functor from the category of rings and ring homomorphisms to the category of topological spaces and continuous maps.

Describe the image of the natural inclusion $\mathbb{R}[x] \to \mathbb{C}[x]$ under the functor Spec.

2 Let A be a ring and S a multiplicatively closed subset of A. Define the localisation A_S of A at S, and prove that it is a ring.

Write down the universal property that characterises A_S and show that it does satisfy this property.

What does it mean to localise A at a prime ideal?

If each localisation of A at a prime ideal has no non-zero nilpotent elements, can A have non-zero nilpotent elements? If each localisation of A at a prime ideal is an integral domain, must A be an integral domain? In each case justify your answer.

3 Let A be a ring. What does it mean to say that an A-module is *Noetherian*? What does it mean to say that A is a *Noetherian ring*?

Define $A[[x]]$ the ring of formal power series with coefficients in A. Show that A is a Noetherian ring if and only if $A[[x]]$ is a Noetherian ring.

Suppose that M is a Noetherian A-module and $f\colon M \to M$ is a surjective A-module map. Show that f is an isomorphism.

4 Define the *Picard group* of a ring. Prove that it is an abelian group.

Show that if $f\colon A \to B$ is a ring homomorphism and L is a line bundle over A then $B \otimes_A L$ is a line bundle over B. Use this to show that Pic defines a functor from the category of rings and ring homomorphisms to the category of abelian groups and group homomophisms.

What is the Picard group of $\mathbb{C}[x]$? Justify your answer.

5 Suppose A and B are rings. Discuss the construction and properties of the left derived functors L_iF of a right exact functor

$$F\colon A\text{–mod} \to B\text{–mod}.$$

Illustrate your discussion with the base change functor $M \mapsto B \otimes_A M$ in the case that $B = A/Aa$ for $a \in A$ not a zero divisor.

6 Let A be a Noetherian ring and M a finitely generated A-module. Show that M has a projective resolution consisting of finitely generated projective A-modules.

Deduce if a module M has projective dimension n then $\mathrm{Ext}^n_A(M, A)$ is non-zero.

Find a ring A and an A-module M such that M does not have finite projective dimension as an A-module. Jusify your answer.

END OF PAPER