

MATHEMATICAL TRIPOS      Part III

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Monday, 8 June, 2009    1:30 pm to 4:30 pm

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**PAPER 29**

**STOCHASTIC CALCULUS AND APPLICATIONS**

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

*The questions carry equal weight.*

**STATIONERY REQUIREMENTS**

*Cover sheet  
Treasury Tag  
Script paper*

**SPECIAL REQUIREMENTS**

*None*

**You may not start to read the questions  
printed on the subsequent pages until  
instructed to do so by the Invigilator.**

**1** (a) Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space. Define the *previsible*  $\sigma$ -algebra  $\mathcal{P}$  and explain what is meant by a *simple process*. (We let  $\mathcal{S}$  be the space of simple processes). Give the definition of the stochastic integral  $H \cdot M$  of a simple process  $(H_s, s \geq 0)$  with respect to a continuous martingale  $M$  which is bounded in  $L^2$ . (We let  $\mathcal{M}_c^2$  be the space of continuous martingales bounded in  $L^2$ ). Give the definition of the quadratic variation  $[M]$  of a continuous local martingale  $M$  and explain how you can compute it from the path  $(M_t, t \geq 0)$  using an approximation procedure. [You are not required to prove the existence of the quadratic variation or to justify your approximation procedure.]

(b) Let  $H \in \mathcal{S}$  and  $M \in \mathcal{M}_c^2$ . Show that  $H \cdot M \in \mathcal{M}_c^2$  and that

$$\mathbb{E}((H \cdot M)_\infty^2) \leq \|H\|_\infty^2 \mathbb{E}((M_\infty - M_0)^2).$$

(c) Show that for  $H \in \mathcal{S}$  and  $M \in \mathcal{M}_c^2$ , we have in fact the equality

$$\mathbb{E}((H \cdot M)_\infty^2) = \mathbb{E}\left(\int_0^\infty H_s^2 d[M]_s\right).$$

Deduce that  $[H \cdot M] = H^2 \cdot [M]$ . [*Hint:* You may use the Optional Stopping Theorem provided that you state it clearly.]

**2** Let  $M$  be a continuous local martingale and let  $A$  be a finite variation process such that for some nonrandom constant  $K < \infty$  we have:

$$\sup_{s \geq 0} (|M_s| + V_s) \leq K \quad (1)$$

where  $V_s$  denote the total variation of  $A$  at time  $s$ . Let  $X_t = M_t + A_t$ , and assume that  $M_0 = A_0 = 0$ .

(a) Let  $\phi$  be a nondecreasing function of class  $\mathcal{C}^1$  such that  $\phi(x) = -1$  for  $x \leq 0$  and  $\phi(x) = 1$  for  $x \geq 1$ . For all  $n \geq 1$ , define a function  $f_n$  such that  $f_n(0) = 0$  and for all  $x \in \mathbb{R}$ ,  $f'_n(x) = \phi(nx)$ . Show that  $f_n(X_t)$  is a continuous semimartingale and give its Doob-Meyer decomposition. Show that as  $n \rightarrow \infty$ ,

$$\int_0^t f'_n(X_s) dM_s \longrightarrow \int_0^t \operatorname{sgn}(X_s) dM_s$$

in the u.c.p. sense (uniformly on compacts in probability), where  $\operatorname{sgn}(x) = 1_{\{x > 0\}} - 1_{\{x \leq 0\}}$  is the (left-continuous) function which gives the sign of  $x$ . [*Hint*: Apply Itô's isometry property to the difference of those two integrals.]

(b) Show that as  $n \rightarrow \infty$ ,

$$\int_0^t f'_n(X_s) dA_s \longrightarrow \int_0^t \operatorname{sgn}(X_s) dA_s$$

almost surely for all  $t \geq 0$  simultaneously.

(c) Deduce from (a) and (b) that if  $Z_t = |X_t| - \int_0^t \operatorname{sgn}(X_s) dX_s$ , then  $Z$  is a nondecreasing process almost surely. Conclude that  $|X|$  is a continuous semimartingale. Show that the result remains true if we no longer assume (??), i.e., if  $X$  is any continuous semimartingale starting at 0.

**3** (a) State the Dubins-Schwartz theorem. Let  $(M^1, M^2)$  be two continuous local martingales in a common filtration  $(\mathcal{F}_t, t \geq 0)$ , such that  $[M^1, M^2]_t = 0$  and  $[M^1]_t = [M^2]_t$ , almost surely for all  $t \geq 0$ . (Here  $[M, N]$  denotes the covariation of the two continuous local martingales  $M$  and  $N$ , and  $[M]$  stands for  $[M, M]$ ). We also assume that if  $A_t = [M^1]_t$ , then  $A_\infty = \infty$  almost surely. Show that if  $\tau_r = \inf\{t \geq 0 : A_t \geq r\}$ , then

$$B_r = (M_{\tau_r}^1, M_{\tau_r}^2), r \geq 0,$$

defines a 2-dimensional Brownian motion in the filtration  $(\mathcal{G}_r)_{r \geq 0}$  defined by  $\mathcal{G}_r = \mathcal{F}_{\tau_r}$  for all  $r \geq 0$ . [You may use any result from the course provided that it is clearly stated.]

(b) Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space such that the filtration  $(\mathcal{F}_t)_{t \geq 0}$  is right-continuous and  $\mathcal{F}_0$  contains all zero probability events. Let  $(\beta_t, t \geq 0)$  and  $(\theta_t, t \geq 0)$  be two independent  $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motions in  $\mathbb{R}$ , and define a process  $(Z_t, t \geq 0)$  with values in the complex plane as follows: let  $R_t = \exp(\beta_t)$ , and let

$$Z_t = R_t e^{i\theta_t}, \quad t \geq 0.$$

Let  $X_t$  and  $Y_t$  denote respectively the real and imaginary parts of  $Z_t$ . Show that  $X$  and  $Y$  are continuous local martingales and that

$$[X]_t = [Y]_t \quad \text{and} \quad [X, Y]_t = 0$$

for all  $t \geq 0$ . Show that  $[X]_\infty = \infty$  almost surely. [*Hint*: the recurrence and the strong Markov property of  $(\beta_t, t \geq 0)$ , together with the law of large numbers, may be helpful for this result.]

(c) Deduce from (a) that there exists  $(\tau_r, r \geq 0)$  such that  $(Z_{\tau_r}, r \geq 0)$  is a two-dimensional Brownian motion started at  $z_0 = (1, 0)$  in an appropriate filtration. Use this to show that Brownian motion in  $\mathbb{R}^2$  started from  $z_0$  almost surely never hits 0 but comes arbitrarily close to 0 on an unbounded set of times.

4

- (a) Let  $M$  be a continuous local martingale and let  $Z = \mathcal{E}(M)$ , where  $\mathcal{E}(M)_t = \exp(M_t - (1/2)[M]_t)$  denotes the exponential local martingale associated with  $M$ . Show that  $dZ_t = Z_t dM_t$ . Suppose now that  $Z$  and  $Z'$  are two strictly positive processes such that  $Z_0 = Z'_0$  and

$$dZ_t = Z_t dM_t; \quad dZ'_t = Z'_t dM_t.$$

By considering  $\ln(Z'_t) - \ln(Z_t)$ , show that  $Z$  and  $Z'$  are indistinguishable.

- (b) Let  $\mathbb{P}$  denote the Wiener measure (i.e., the law of a Brownian motion  $(X_t, t \geq 0)$ ), and let  $\mathbb{Q}$  be a measure which is absolutely continuous with respect to  $\mathbb{P}$  on  $\mathcal{F}_t$  for every  $t > 0$ , where  $(\mathcal{F}_t)_{t \geq 0}$  is the filtration generated by  $X$ . We denote by  $Z_t$  the Radon-Nikodym derivative of  $\mathbb{Q}$  with respect to  $\mathbb{P}$  on  $\mathcal{F}_t$ : that is,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = Z_t.$$

Show that  $(Z_t, t \geq 0)$  is a martingale in  $(\mathcal{F}_t)_{t \geq 0}$ . Assume that  $Z_t = h(X_t, t)$ , where  $h(x, t)$  is a given positive  $\mathcal{C}^2$  function. Deduce that  $D_t h + \frac{1}{2} D_{xx} h = 0$ , where  $D_t h$ ,  $D_{tt} h$  (resp.  $D_x h$ ,  $D_{xx} h$ ) denote the first and second derivatives of  $h$  with respect to  $t$  (resp.  $x$ ).

- (c) Using the same notations as in (b), show that  $dZ_t = Z_t dM_t$  where  $M$  is defined by:

$$M_0 = 0; \quad dM_t = \frac{D_x h(X_t, t)}{h(X_t, t)} dX_t.$$

Deduce from this and the result in (a) that  $Z_t = \mathcal{E}(M)_t$ . [You may use without proof the fact that  $Z_0 = 1$  almost surely].

Conclude that the semimartingale decomposition of  $X$  under  $\mathbb{Q}$  is

$$X_t = B_t + \int_0^t \frac{D_x h(X_s, s)}{h(X_s, s)} ds,$$

where  $B$  is a Brownian motion. [*Hint:* You may use Girsanov's theorem without proof, provided that you state it clearly and verify the assumptions carefully].

## 5

(a) Let  $(B_t, t \geq 0)$  be a one-dimensional Brownian motion and let  $Y_t = e^{B_t+t/2}$ . Show that  $Y$  solves a certain stochastic differential equation, whose coefficients should be determined. Does pathwise uniqueness hold for this equation? Using the Dubins-Schwartz theorem, show that there exists a Brownian motion  $(\beta_t, t \geq 0)$  such that

$$Y_t = 1 + \beta_{H_t} + \int_0^t \frac{1}{Y_s} dH_s \quad (1)$$

where  $H_t = \int_0^t e^{2B_s+s} ds$ .

(b) The notations are the same as in (a). Let  $J_t = \inf\{s > 0 : H_s > t\}$ . Deduce from the above that if  $X_t = Y_{J_t}$ , then

$$X_t = 1 + \beta_t + \int_0^t \frac{1}{X_s} ds. \quad (2)$$

[*Hint*: note that  $H_{J_t} = t$  for all  $t \geq 0$  and use a change of variable in the integral appearing in the right-hand side of (??).]

(c) Let  $(W_t, t \geq 0)$  be a Brownian motion in  $\mathbb{R}^3$  started from  $W_0 = (1, 0, 0)$ , and let  $|W_t|$  denote the Euclidean norm of  $W_t$ . Show that  $|W|$  is also a solution of (??) and deduce that  $\lim_{t \rightarrow \infty} |W_t| = \infty$  almost surely. [You may assume without proof that there is uniqueness in distribution for the solutions of (??).]

Show further that

$$\lim_{t \rightarrow \infty} \frac{\log(|W_t|)}{\log(t)} = \frac{1}{2},$$

almost surely. Assuming without proof the existence of this limit, explain briefly how you could have guessed its value using a different method.

## 6

Throughout this question, we fix a probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  satisfying the *usual conditions*: the filtration  $(\mathcal{F}_t)_{t \geq 0}$  is right-continuous and  $\mathcal{F}_0$  contains all events of probability zero.

(a) Let  $\sigma, b : \mathbb{R} \rightarrow \mathbb{R}$  be two measurable and locally bounded functions, and suppose that an adapted continuous stochastic process  $(X_t, t \geq 0)$  is a solution to the stochastic differential equation:

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt \quad (1)$$

where  $(B_t, t \geq 0)$  is a one-dimensional  $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion. Assume that  $s : I \rightarrow \mathbb{R}$  is a  $C^2$  function on an interval  $I$  such that

$$\frac{1}{2}s''(x)\sigma^2(x) + s'(x)b(x) = 0, \quad x \in I. \quad (2)$$

Show that  $Y_t = s(X_{t \wedge T})$  is a local martingale, where  $T = \inf\{t \geq 0 : X_t \notin I\}$ . Such a function  $s$  is called a scale function for (??) on  $I$ . Deduce that if  $a < x < b$  with  $a, b \in I$ , and  $X_0 = x$  almost surely, then

$$\mathbb{P}(T_b < T_a) = \frac{s(x) - s(a)}{s(b) - s(a)},$$

where for all  $y \in \mathbb{R}$ ,  $T_y = \inf\{t \geq 0 : X_t = y\}$ .

(b) Let  $a > 0$  with  $a \neq 1/2$ , and assume that  $X$  is a positive solution to the stochastic differential equation

$$dX_t = dB_t + \frac{a}{X_t}dt. \quad (3)$$

Show that for all  $\epsilon > 0$ ,  $s(x) = x^{-2a+1}$  is a scale function for (??) on  $[\epsilon, \infty)$ . Conclude that for all  $x > 0$ , if  $X_0 = x$ , then  $T_0 = \infty$  almost surely if  $a > 1/2$ , while  $T_0 < \infty$  almost surely if  $a < 1/2$ .

(c) Assume that  $a > 1/2$ . Show that for every  $\epsilon > 0$  and for every driving Brownian motion  $B$ , there exists a unique process  $(X_t^\epsilon, t \geq 0)$  which satisfies (??) for all  $t < T_\epsilon^\epsilon$ , where for all  $\epsilon > 0$ ,  $T_\epsilon^\epsilon = \inf\{t \geq 0 : X_t^\epsilon = \epsilon\}$ . Show that  $T_\epsilon^\epsilon$  is nondecreasing as  $\epsilon \rightarrow 0$ , and let  $T = \lim_{\epsilon \rightarrow 0} T_\epsilon^\epsilon$ . Deduce that one can construct a process  $(X_t, t \geq 0)$  which is a solution of (??) for all  $t < T$ . Show that necessarily  $T = \infty$  almost surely. Conclude that in the case  $a > 1/2$  there exists a strong solution to (??) for every driving Brownian motion  $B$  and that pathwise uniqueness holds.

**END OF PAPER**