UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2009 1:30 pm to 4:30 pm

PAPER 28

LOCAL FIELDS

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) State and prove a version of Hensel's Lemma.

(ii) Let $f(x) = x^3 - 3x + 4$. Show that the equation $f(x) = 0$ has a unique solution in $\mathbb{Z}_7$ but no solutions in $\mathbb{Z}_5$ or $\mathbb{Z}_3$. Find how many solutions it has in $\mathbb{Z}_2$.

(iii) Let $f\colon \mathbb{Z}_2 \to \mathbb{Z}_2$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in 2\mathbb{Z}_2 \\ -1 & \text{if } x \notin 2\mathbb{Z}_2 \end{cases}$$

Show that f is continuous and compute its Mahler expansion.

2 Let L/K be a finite extension of local fields. Show that L/K is unramified if and only if $L = K(x)$ for some $x \in \mathfrak{o}_L$ whose minimal polynomial g satisfies $g'(x) \not\equiv 0 \pmod{\mathfrak{m}_L}$.

Suppose that L/K is unramified, and M/K is arbitrary. Show that every k_K-homomorphism $k_L \to k_M$ is induced by a unique K-homomorphism $L \to M$. Deduce that L/K is Galois with cyclic Galois group.

Let L/K be a finite unramified extension, and M/K a totally ramified extension, both contained in a fixed algebraic closure of K. Show that the field LM is totally ramified over L.

Find an example of a finite extension $N/\mathbb{Q}_p$ which is not of the form $N = LM$ for an unramified extension $L/\mathbb{Q}_p$ and a totally ramified extension $M/\mathbb{Q}_p$.

3 Let L/K be a finite extension of local fields. Show that L/K is totally ramified if and only if $L = K(x)$ for some $x \in \mathfrak{o}_L$ whose minimal polynomial is an Eisenstein polynomial, and that in that case x is a uniformiser of L.

Show that if q is a power of p then $\mathbb{Q}_p(\zeta_q)$ is totally ramified. For what other values of n is it the case that $\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p$ is totally ramified?

If L/K is finite, Galois and totally ramified, define the *ramification groups* of L/K. Determine them for the extension $\mathbb{Q}_p(\zeta_q)/\mathbb{Q}_p$.

4 (i) Prove that if L/K is an unramified extension of local fields of degree n then $N_{L/K}(L^*) = \{x \in K^* \mid v_K(x) \equiv 0 \mod n\}$.

(ii) State and prove Hilbert's Theorem 90. Deduce that if L/K is a finite unramified extension of local fields with Frobenius $\phi_{L/K}$ then for every $x \in \mathfrak{o}_L^*$ with $N_{L/K}(x) = 1$ there exists $y \in \mathfrak{o}_L^*$ with $\phi_{L/K}(y)/y = x$.

5 Write an essay on the reciprocity map of local class field theory. You should include a definition of the reciprocity map and a statement of its properties, with an outline of some of the proofs.

END OF PAPER