UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2009 9:00 am to 12:00 pm

PAPER 27

MODULAR FORMS

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $k \geqslant 2$ be an integer.

(a) Explain what is meant by a *normalised eigenform* in $S_k(\mathrm{SL}_2(\mathbb{Z}))$. Show that if f is a normalised eigenform and $\lambda(m)$ is the T_m-eigenvalue of f, then we have

$$\begin{cases} \lambda(mn) = \lambda(m)\lambda(n) & \text{for } (m,n) \text{ coprime} \\ \lambda(p^{e+2}) = \lambda(p)\lambda(p^{e+1}) - p^{k-1}\lambda(p^e) & \text{for } p \text{ prime, } e \geqslant 0. \end{cases}$$

[*You should give proofs of any statements you use concerning Hecke operators.*]

(b) Show that if n is prime, then the T_n-eigenvalue $\lambda(n)$ of f is equal to $a_n(f)$, the coefficient of q^n in the q-expansion of f.

(c) By considering the values $\lambda(p^e)$ for p a fixed prime, or otherwise, show that there exist infinitely many n such that $\lambda(n) \geqslant \frac{1}{2}n^{(k-1)/2}$. [*You may assume the polynomial* $X^2 - \lambda(p)X + p^{k-1}$ *has distinct roots for all* p.]

2 Let $n \geqslant 0$ be an integer, Γ a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and $f \in \mathcal{A}_{2n}(\Gamma)$. Show that the meromorphic differential $\lambda(f) = f \cdot (\mathrm{d}z)^n \in \Omega^n(\mathcal{H})$ is Γ-invariant.

Construct a meromorphic differential $\omega(f) \in \Omega^n(X(\Gamma))$ such that $\pi_\Gamma^*[\omega(f)] = \lambda(f)$, and derive formulae relating the orders of vanishing of f and $\omega(f)$.

Let k and N be positive integers and χ a Dirichlet character modulo N. Suppose there exists some nonzero $f \in \mathcal{A}_k(\Gamma_1(N), \chi)$. Construct a divisor $D(f)$ on $X_0(N)$ for which

$$S_k(\Gamma_1(N), \chi) = \{f\phi : \phi \in \mathcal{L}(D(f))\}.$$

By considering $D(fg^t)$ where g is the element of $\mathcal{A}_2(\Gamma_0(N))$ corresponding to a nonzero meromorphic 1-differential, or otherwise, show that there is some t such that $S_{k+2t}(\Gamma_1(N), \chi) \neq 0$. Show moreover that

$$\dim S_{k+2t}(\Gamma_1(N), \chi) = \frac{dt}{6} + A(t)$$

where $d = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$ and $A(t)$ depends only on t mod 6.

3

(a) Show that the group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the elements $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Describe the standard fundamental domain $\mathcal{D}$ for $\mathrm{SL}_2(\mathbb{Z})$ and show that every $\mathrm{SL}_2(\mathbb{Z})$-orbit in $\mathcal{H}$ contains a point of $\mathcal{D}$.

(b) Show that every modular form of level $\mathrm{SL}_2(\mathbb{Z})$ may be written as a polynomial in E_4 and E_6. Show that if N is an integer and $f = \sum_{n\geqslant 0} a_n q^n \in M_k(\mathrm{SL}_2(\mathbb{Z}))$, and $a_n \in \mathbb{Z}$ for $0 \leqslant n \leqslant \frac{k+1}{12}$, then $a_n \in \mathbb{Z}$ for all n. [*You may assume that the unique normalised cusp form* $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ *has integral coefficients.*]

(c) Let λ be an element of $\mathbb{R}$ with $0 < \lambda < 1$. The subgroup $\Gamma_\lambda \subseteq \mathrm{SL}_2(\mathbb{R})$ is generated by the matrices

$$A = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \text{ and } B = \begin{pmatrix}1 & 2\lambda\\ 0 & 1\end{pmatrix}.$$

By considering the stabiliser of the point $z = -\lambda + i\sqrt{1-\lambda^2}$, or otherwise, show that if Γ_λ is discrete, we must have $\lambda = \cos \pi y$ for some $y \in \mathbb{Q}$.

4

(a) Let $N \geqslant 1$ be an integer and ℓ a prime not dividing N. By applying the matrix identity

$$\begin{pmatrix}\ell & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & n\\ N & m\ell\end{pmatrix}^{-1} \begin{pmatrix}1 & 0\\ 0 & \ell\end{pmatrix} \begin{pmatrix}\ell & n\\ N & m\end{pmatrix}$$

for suitable m, n, or otherwise, show that in $\mathcal{R}(\Gamma_1(N))$ we have $w_N T_\ell w_N^{-1} = \langle \ell^{-1}\rangle T_\ell$, and T_ℓ is a normal operator on $S_k(\Gamma_1(N))$ with respect to the Petersson product.

(b) Show that $S_k(\Gamma_1(N))^{\mathrm{new}}$ has a basis of eigenforms for all the Hecke operators T_ℓ ($\ell \nmid N$) and U_p ($p \mid N$), and

$$\left\{ f(tz) \;\middle|\; \begin{array}{c} f \in S_k(\Gamma_1(M))^{\mathrm{new}} \text{ primitive eigenform} \\ M \mid N, t \mid \frac{N}{M} \end{array} \right\}$$

is a basis for $S_k(\Gamma_1(N))$. Show that if $f \in S_k(\Gamma_0(N))^{\mathrm{new}}$ is a primitive eigenform, $w_N f = \pm N^{k-1} f$. [*You may assume that the Hecke operators preserve the new subspace.*]

(c) Let f be a primitive eigenform in $S_k(\Gamma_0(M))^{\mathrm{new}}$, p a prime dividing M, and $N = Mp^r$ for some integer $r \geqslant 1$. Give matrices for the action of U_p and w_N on the subspace of $S_k(\Gamma_0(N))^{\mathrm{old}}$ corresponding to f, and hence show that this space is a *simple Hecke module*, i.e. it has no nonzero proper subspace that is invariant under $\mathcal{R}(\Gamma_0(N))$.

5 Write an essay on the theory of modular symbols, explaining how they may be used to prove that the matrices of Hecke operators on weight 2 cusp forms are rational and algorithmically computable.

Illustrate the theory by reference to the (unique) subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ with the property that $\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_{i=1}^{7} \Gamma r_i$, for some elements $r_1, \ldots, r_7$, with $\Gamma r_1 = \Gamma$, and right multiplication by the elements $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ induce the permutations $(1,7,3,6)(2,5,4)$ and $(1,7)(2,6)(3,4)$ of the cosets $\{\Gamma r_i\}$. What is the rank of $H_1(X(\Gamma), \mathbb{Z}, \{\text{cusps}\})$ in this case?

END OF PAPER