UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2009 1:30 pm to 4:30 pm

PAPER 24

TOPOS THEORY

*Attempt **TWO** questions from **Section I** and **ONE** question from **Section II**.*

*There are **SEVEN** questions in total.*

The questions carry equal weight.

The paper is in two sections, Section I comprising the pure bookwork questions 1–4 and Section II the questions 5–7 which have a problem element.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1 Starting from the definition of a (weak) topos, prove that the contravariant power-object functor is monadic. Deduce that a topos has products for families of objects of cardinality κ if and only if it has coproducts for such families.

[Standard theorems on monadicity, and on lifting of adjoints, may be assumed.]

2 Show that if $\mathbb{G}$ is a cartesian comonad on a topos $\mathcal{E}$, then the category $\mathcal{E}_{\mathbb{G}}$ of $\mathbb{G}$-coalgebras is a topos. Deduce that any geometric morphism may be factored as a surjection followed by an inclusion, and that the factorization is unique up to equivalence.

3 Define the notions of closed and quasi-closed local operator, and show that a subtopos $\mathbf{sh}_j(\mathcal{E})$ of $\mathcal{E}$ is Boolean if and only if j is quasi-closed. Use quasi-closed local operators to show that every topos admits a surjection from a Boolean topos.

4 Explain what is meant by a coherent theory over a signature Σ, and sketch the construction of the syntactic category $\mathcal{C}_{\mathbb{T}}$ of a coherent theory $\mathbb{T}$. Show that, for any topos $\mathcal{E}$, the category of $\mathbb{T}$-models in $\mathcal{E}$ is equivalent to the category of coherent functors $\mathcal{C}_{\mathbb{T}} \to \mathcal{E}$.

SECTION II

5 Define the notion of coverage on a small category $\mathcal{C}$, and explain what it means for a coverage to be rigid. If $\mathcal{C}$ is a finite category in which all idempotents split, show that any coverage J on $\mathcal{C}$ is rigid, and deduce that $\mathbf{Sh}(\mathcal{C}, J)$ is equivalent to a functor category $[\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$.

6 If U is a subobject of 1 in a topos $\mathcal{E}$, the *open local operator* $o(U)$ is defined to be $u \Rightarrow (-)\colon \Omega \to \Omega$, where $u\colon 1 \to \Omega$ is the classifying map of U. Verify that $o(U)$ is a local operator, and show that its category of sheaves is equivalent to $\mathcal{E}/U$. [Hint: first show that a monomorphism m is $o(U)$-dense iff $U^*(m)$ is an isomorphism.] Show also that an arbitrary local operator j is open if and only if the dense-monomorphism classifier J has a least element, i.e. there exists a morphism $1 \to J$ such that the pair $(J \to 1 \to J, 1_J)$ factors through the order relation.

7 Give a presentation of the theory of local (commutative) rings as a coherent theory, and sketch the proof that its classifying topos may be identified with $\mathbf{Sh}(\mathcal{C}, Z)$ where $\mathcal{C}$ is the dual of the category of finitely-presented rings and Z is the Zariski coverage. Given that this site is standard, show that the generic local ring R satisfies the 'Kock–Lawvere axiom' that R^D is isomorphic to $R \times R$, where $D = [\![x.(x^2 = 0)]\!]_R$.

END OF PAPER