UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2009 1:30 pm to 4:30 pm

PAPER 23

CATEGORY THEORY

Attempt no more than ***FIVE*** *questions.*

There are ***EIGHT*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Prove that a category that has finite products and equalisers has all finite limits.

(b) Prove that a category with pullbacks and a terminal object has binary products and equalisers; hence deduce that it has all finite limits.

2 What does it mean for a functor to *preserve* limits of shape $\mathcal{I}$? What does it mean for a functor to *reflect* limits of shape $\mathcal{I}$? Show that a full and faithful functor reflects limits.

(a) Suppose there are given functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$. Show that if GF preserves limits of shape $\mathcal{I}$ and G reflects them, then F preserves limits of shape $\mathcal{I}$.

(b) If X is an object of $\mathcal{D}$, show that the forgetful functor $U_X\colon \mathcal{D}/X \to \mathcal{D}$ from the slice category reflects pullbacks and preserves equalisers.

(c) Let $\mathcal{C}$ be a category with all finite limits and $F\colon \mathcal{C} \to \mathcal{D}$ a pullback-preserving functor. By factorising F as

$$\mathcal{C} \xrightarrow{\hat{F}} \mathcal{D}/F1 \xrightarrow{U_{F1}} \mathcal{D}$$

where 1 is the terminal object of $\mathcal{C}$ and $\hat{F}(A)$ the image under F of the unique morphism $A \to 1$, show that F preserves equalisers. *[You may assume that a functor which preserves pullbacks and the terminal object preserves all finite limits.]*

3 Let $\mathcal{C}$ be a small category.

(a) Prove that every $F \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is a colimit of representable presheaves.

(b) Define *cartesian closed category*, and prove that $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is cartesian closed. *[You may assume that limits and colimits are computed pointwise in a presheaf category.]*

(c) Suppose that $\mathcal{C}$ is a small cartesian closed category. Show that the Yoneda embedding $H_\bullet\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ preserves exponentials, in the sense that $(H_Z)^{H_Y} \cong H_{(Z^Y)}$.

4 Let $\mathcal{C}$ be a small category.

(a) Define the *Yoneda embedding* $H_\bullet\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. State the Yoneda Lemma and use it to deduce that $H_\bullet$ is full and faithful.

A *retract diagram* is a pair of maps

$$A \xrightarrow{\;i\;} B \xrightarrow{\;p\;} A \tag{1}$$

such that $pi = \mathrm{id}_A$.

(b) Show that in a retract diagram like (**??**), $i\colon A \to B$ is the equaliser of the maps ip and $\mathrm{id}_B\colon B \to B$.

(c) Show that if $\mathcal{C}$ has equalisers, then for any B in $\mathcal{C}$ and any retract diagram

$$F \xrightarrow{\;i\;} H_B \xrightarrow{\;p\;} F$$

in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, the presheaf F is representable. *[You may assume standard results about the Yoneda embedding.]*

An object A of a category $\mathcal{E}$ is said to be 0-*presentable* if the functor $\mathcal{E}(A,-)\colon \mathcal{E} \to \mathbf{Set}$ preserves colimits. In elementary terms, this says that any map from A into the vertex of a colimiting cone in $\mathcal{E}$ will factor through one of the colimit injections in an essentially unique way.

(d) Show that every representable functor in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is 0-presentable. *[You may assume standard results about colimits in presheaf categories.]*

(e) Show that if $\mathcal{C}$ has equalisers, then every 0-presentable object of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is representable. *[Hint: express the object in question as a colimit of representables.]*

5 Explain how a poset is realised as a category. What is a functor between posets?

(a) Let $f\colon X \to Y$ and $g\colon Y \to X$ be a pair of functors between posets. Describe in explicit terms what an adjunction $f \dashv g$ is.

(b) Let $p\colon A \to B$ be a map of sets and let $p^*\colon \mathcal{P}B \to \mathcal{P}A$ be the induced map of power-sets sending $X \subseteq B$ to $p^*(X) = \{\, a \in A \mid p(a) \in X \,\}$. Exhibit left and right adjoints to p^*.

(c) Fix a non-empty topological space S, and let $\mathcal{O}(S)$ denote the poset of open subsets of S, ordered by inclusion. Let

$$\Delta\colon \mathbf{Set} \to [\mathcal{O}(S)^{\mathrm{op}}, \mathbf{Set}]$$

be the functor assigning to a set the presheaf ΔA with constant value A. Exhibit left and right adjoints to Δ. *[In this part you are only expected to define the functors on objects and when you show adjointness you are not expected to carry out any formal checks of naturality.]*

6 State and prove the General Adjoint Functor Theorem. (If you wish to appeal to an initial object lemma, you should prove it.)

7

(a) Define the structure of a *monad* on a category $\mathcal{C}$, and the *category of algebras* for a monad.

(b) Show that any adjunction $F \dashv G\colon \mathcal{D} \to \mathcal{C}$ induces a monad on $\mathcal{C}$.

(c) Show that the forgetful functor from the category $\mathcal{C}^{\mathbb{T}}$ of $\mathbb{T}$-algebras to $\mathcal{C}$ has a left adjoint.

8 Let $\mathbb{T}$ be a monad on a category $\mathcal{C}$ induced by an adjunction $F \dashv G \colon \mathcal{D} \to \mathcal{C}$.

(a) Define the *comparison functor* $K \colon \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$. *[You need not prove that it is well-defined.]*

(b) Give, with proof, necessary and sufficient conditions for K to have a left adjoint.

(c) Give, with proof, necessary and sufficient conditions for the adjunction in part (b) to have an invertible unit.

END OF PAPER