UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2009 9:00 am to 12:00 pm

PAPER 22

SPECTRAL GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 For a Riemannian manifold (M,g) define the isomorphisms $\flat : \chi(M) \longrightarrow \Omega^1(M)$ and $\# : \Omega^1(M) \longrightarrow \chi(M)$ between vector fields and 1-forms on M. Also define the trace of a 2-form β and hence define grad$f$ and the Laplacian Δf for a smooth function f on M, taking div$X$ to be trace $D(X^\flat)$.

If $\pi : (M,g) \longrightarrow (N,h)$ is a Riemannian submersion with totally geodesic fibres, establish a relation between the eigenspaces of the Laplacians of M and N acting on smooth functions.

Define a flat torus and describe its spectrum. Given isospectral non-isometric flat tori of dimension four, prove the existence of such a pair of tori in all dimensions $d \geqslant 4$.

2 Describe, with proof, a 3-parameter family of pairs of planar domains such that the members of each pair are not isometric but are isospectral for the Laplacian acting on smooth functions with the Dirichlet boundary condition.

3 Define the heat kernel, heat trace and heat invariants of a Riemannian manifold. Assuming the existence of the heat kernel for a compact Riemannian manifold, prove its uniqueness.

Deduce that the heat trace determines the spectrum of the Laplacian and *vice versa* and that isospectral manifolds have the same dimension.

4 Given a finitely presented group T, describe how to construct a compact manifold M without boundary such that $\pi_1(M) \cong T$. You may quote without proof any results from differential topology, including the theory of handle decompositions, that you require.

If T is finite and U_1 and U_2 are Gassman equivalent subgroups, show how to construct isospectral Riemannian manifolds M_1 and M_2 with $\pi_1(M_i) \cong U_i$, $i = 1, 2$.

If U_1 and U_2 are conjugate in T, prove that M_1 and M_2 are isometric.

Construct isospectral Riemannian manifolds M_1 and M_2 with $\pi_1(M_1)$ not isomorphic with $\pi_1(M_2)$.

5 Given a finite group T generated by two elements, a, b, describe how to construct an oriented closed compact topological surface M on which T acts as a group of orientation preserving homeomorphisms. Identify the stabilisers of this action and find the Euler characteristic $\chi(M)$.

If U is a subgroup of T what is the criterion for the covering $M \longrightarrow U\backslash M$ to be normal and what then is $\chi(U\backslash M)$?

Given a group T with $|T| = 96$ having non-conjugate, Gassman equivalent subgroups U_1, U_2 of index 12 and generators a, b, of order 3 with $(ab)^3$ central of order 2 and not in U_1 or U_2, construct isospectral non-isometric Riemannian manifolds of dimension 2 and genus 2.

END OF PAPER