UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 3 June, 2009 9:00 am to 12:00 pm

PAPER 21

ALGEBRAIC GEOMETRY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 This problem goes through the main part of the proof that projective space $\mathbf{P}^n$ over the field k is compact. That is, for any algebraic set X over k and any closed subset $Y \subset X \times \mathbf{P}^n$, the image of Y under the projection $\pi : X \times \mathbf{P}^n \to X$ is closed in X. (Also, $\mathbf{P}^n$ is separated, but we won't check that here.)

(a) Show that it suffices to prove the above statement for X affine.

(b) Let $f_1, \ldots, f_r$ be homogeneous polynomials in $k[y_0, \ldots, y_n]$. Let $\mathbf{m}$ be the ideal $(y_0, \ldots, y_n)$. Show that the set $\{f_1 = 0, \ldots, f_r = 0\} \subset \mathbf{P}^n$ is empty if and only if the ideal $(f_1, \ldots, f_r)$ contains $\mathbf{m}^N$ for some $N \geqslant 0$. [*You may use general theorems on affine algebraic geometry.*]

(c) Show that

$$\pi(Y) = \bigcap_{N \geqslant 0} \{x \in X : \text{the ideal in } k[y_0, \ldots, y_n] \text{ generated by } f_1(x,y), \ldots, f_r(x,y) \text{ does not contain } \mathbf{m}^N\}.$$

[*For X affine, say X closed in A^m, you may use that any closed subset of $X \times \mathbf{P}^n$ is given by the vanishing of some polynomials $f_1(x,y), \ldots, f_r(x,y)$ (writing x for $x_1, \ldots, x_m$ and y for $y_0, \ldots, y_n$) which are homogeneous in y.*]

(d) Show that, for any algebraic set X over k and any closed subset $Y \subset X \times \mathbf{P}^n$, the image $\pi(Y) \subset X$ is closed in X.

2 (a) Let X be a smooth projective curve of genus g over k, and let $X(k)$ be its set of k-points. Let $Pic^0(X)$ be the group of line bundles of degree 0 on X, and write $O(D)$ for the line bundle corresponding to a divisor D. Fix a point $p_0 \in X(k)$. Show that, if $g \geqslant 1$, then the function $\alpha : X(k) \to Pic^0(X)$ defined by $\alpha(p) = O(p - p_0)$ is injective. Also, what is $Pic^0(X)$ if $g = 0$?

(b) For X of genus 1, use part (a) to define the structure of an abelian group on the set $X(k)$, by showing that α is bijective.

(c) Show that, for X of any genus, the abelian group $Pic^0(X)$ is generated by the subset $\alpha(X(k)) \subset Pic^0(X)$.

3 Give examples, with justification, of:

(a) a smooth curve of degree 3 over the algebraic closure of the field $\mathbf{F}_3 = \mathbf{Z}/3$.

(b) an irreducible surface in $\mathbf{P}^3$ over $\mathbf{C}$ with exactly one singular point.

(c) an irreducible curve in $\mathbf{P}^2$ over $\mathbf{C}$ with exactly two singular points.

4 Let k be an algebraically closed field of characteristic zero. Let the group $\mathbf{Z}/2$ act on the affine plane A^2 over k by $(x, y) \mapsto (-x, -y)$. Let R be the sub-k-algebra of $O(A^2) = k[x, y]$ consisting of the regular functions which are constant on all $\mathbf{Z}/2$-orbits in A^2. Give a basis for R as a k-vector space. Show that R is a finitely generated k-algebra which is an integral domain. Let Y be the corresponding affine variety. Describe an embedding of Y into A^3, and find an equation satisfied by Y.

The inclusion $R \subset k[x, y]$ corresponds to a morphism $f : A^2 \to Y$. Show that f is surjective. Finally, find the singular set of Y (the subset where Y is not smooth over k).

5 (a) Show that any regular function on a projective variety is constant.

(b) Let C be a smooth compact curve, $S \subset C$ a finite subset. Show that any morphism from $C - S$ to projective space $\mathbf{P}^n$ extends to a morphism on all of C.

END OF PAPER