UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2009 9:00 am to 12:00 pm

PAPER 20

FOUR-MANIFOLDS

Attempt no more than ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Figure 1a shows a Kirby diagram for a 4-manifold W composed of a 0-handle, two 1-handles, and a 2-handle. Compute $\pi_1(W), H_*(W)$, and $H_*(\partial W)$.

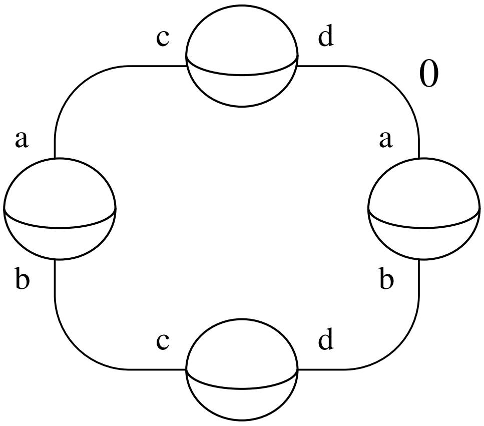

Figure 1a

(ii) Same question for the Kirby diagram of Figure 1b, which consists of a 0-handle and three 2-handles.

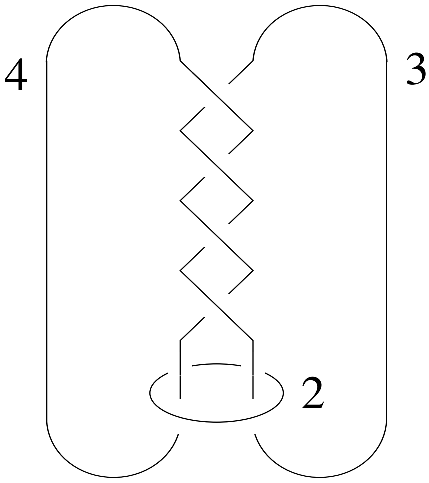

Figure 1b

(iii) For the diagram of Figure 1a, explain how to construct an embedded torus representing a nontrivial class in $H_2(W)$.

(iv) Explain how to construct a negative definite four-manifold bounding $L(7,3)$. What is its intersection form? [*Hint: Consider a Kirby diagram in the form of a chain.*]

2 (i) Let $(\Sigma, \alpha, \beta, z)$ be the pointed genus 1 Heegaard diagram shown in Figure 2a. Find all generators of the complex $CF^-(\Sigma, \alpha, \beta, z)$. Divide them into equivalence classes (where $\mathbf{x} \sim \mathbf{y}$ if and only if $\pi_2(\mathbf{x}, \mathbf{y}) \neq \emptyset$). Compute the relative homological gradings in each equivalence class.

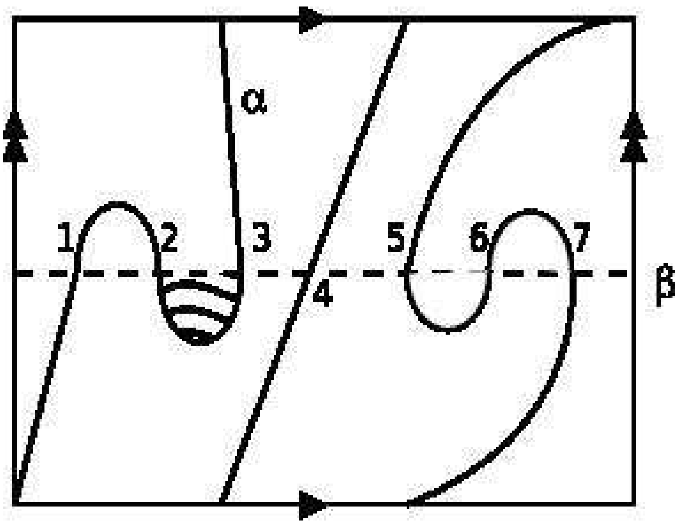

Figure 2a

(ii) Same as part (i), but for the genus 2 Heegaard diagram shown in Figure 2b.

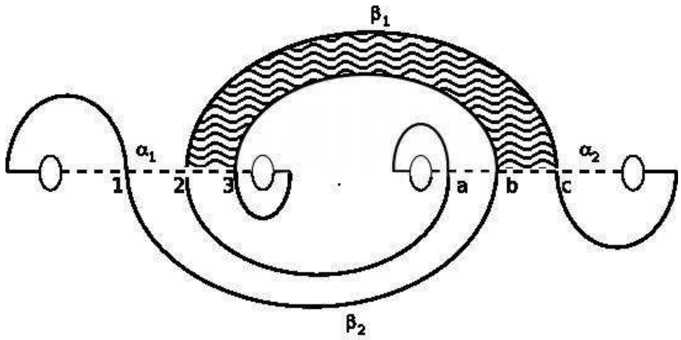

Figure 2b

(iii) Explain why the shaded domains in each diagram have $\#\overline{\mathcal{M}}(\psi) = \pm 1$. Use this to compute the homology of the chain complexes in parts (i) and (ii).

3 (i) Suppose $K \subset S^3$ is a knot, and let K_n denote the result of n-surgery on K. For n a positive integer and $\mathfrak{t}_k$ a $Spin^c$ structure on K_n, state the exact triangle relating $HF^-(K_0)$ to $HF^-(K_n, \mathfrak{t}_k)$. Which maps in the triangle are induced by cobordisms? For those that are, describe the set of $Spin^c$ structures on the cobordism and their restriction to the boundary.

(ii) Now suppose that $K = U$ is the unknot in S^3. Describe the exact triangle of part (i) explicitly in this case. What are the three-manifolds appearing in the triangle, and what is their Floer homology? What are the maps induced by cobordisms? How do they affect the absolute grading?

4 (i) Suppose X is a smooth complex surface and $\Sigma \subset X$ is a smooth complex curve. Derive the adjunction formula relating $c_1(TX)$ and the genus of Σ.

(ii) If Σ is a smooth complex curve of genus 3 in a K3 surface, what is its self-intersection?

(iii) Suppose that M is a closed smooth 4-manifold with $b_2^+ \geqslant 3$ and that Σ is a smoothly embedded surface in M with $\Sigma \cdot \Sigma \geqslant 0$. Suppose further that $\mathfrak{s}$ is a $Spin^c$ structure on M with $\Phi(M, \mathfrak{s}) \neq 0$. Given that $HF^-(\Sigma \times S^1, \mathfrak{t}) = 0$ whenever $\langle c_1(\mathfrak{t}), [\Sigma] \rangle > 2g(\Sigma) - 2$, show that $\langle c_1(\mathfrak{s}), [\Sigma] \rangle + \Sigma \cdot \Sigma \leqslant 2g(\Sigma) - 2$.

5 (i) Let Y be a 3-manifold with $b_1(Y) = 0$ and $\mathfrak{s} \in Spin^c(Y)$. Describe (without proof) the structure of $HF^\infty(Y, \mathfrak{s})$. By considering the structure of $HF^-(Y, \mathfrak{s})$ as a $\mathbb{Z}[U]$ module, or otherwise, show that the Euler characteristic of $\widehat{HF}(Y, \mathfrak{s})$ is always ± 1.

(ii) Suppose $W : Y_1 \to Y_2$ is a cobordism, $\mathfrak{s} \in Spin^c(W)$, and $b_1(Y_i) = 0$. Describe (without proof) the map $\Phi^\infty_{W,\mathfrak{s}} : HF^\infty(Y_1, \mathfrak{s}|_{Y_1}) \to HF^\infty(Y_2, \mathfrak{s}|_{Y_2})$.

(iii) If M is a closed 4-manifold with $b_2^+(M) \geqslant 2$, explain how to define the invariant $\Phi(M, \mathfrak{s})$. Why is the condition on b_2^+ necessary?

END OF PAPER