UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2009 9:00 am to 12:00 pm

PAPER 2

TOPICS IN GROUP THEORY

There are two sections, of three questions each.

Attempt no more than ***THREE*** *questions,* ***NOT*** *all from the same section.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 State and prove the Jordan - Hölder Theorem concerning the composition series of finite groups.

Let G be a group of order 120, with composition factors C_2 and A_5. Show that if G has a normal subgroup K of order 2, then $K = Z(G)$, the centre of G. Deduce that S_5 does not have a normal subgroup of order 2.

Find, with justification, all composition series of the groups $C_2 \times S_5$ and $S_5 \times S_5$.

Show that $SL_2(5)$ has centre of order 2. By considering elements of order 2, show that $SL_2(5)$ has no subgroup of index 2.

2 State the theorem of P. Hall concerning the subgroups of finite soluble groups, and outline briefly its proof.

Let A and B be maximal subgroups in the finite soluble group G. Prove that either G factorises as $G = AB$, or A and B are conjugate in G. You should consider first the case where G has a non-trivial normal subgroup K, contained in A or B; in the other case, consider normal subgroups $K < L$ of G with K minimal normal in G and L/K minimal normal in G/K, and the intersections of A and B with these.

3 State Sylow's Theorems.

Let $G = S_6$, let N be a subgroup of order 20. Show that N is the normaliser of a Sylow 5-subgroup P of G, and that N is contained in the stabiliser H of a point in the natural action of G of degree 6.

By considering the action of S_5 on the set of its Sylow 5-subgroups, show that H is isomorphic to a transitive subgroup of S_6. Deduce the existence of an outer automorphism τ of S_6 that interchanges the conjugacy class of transitive subgroups isomorphic to S_5 with the conjugacy class of point–stabilisers in the natural action of S_6.

Let $\overline{G}$ be the automorphism group of S_6, so that $\overline{G}$ is the group of order 1440 generated by G and τ. Show that the normaliser $\overline{N}$ in $\overline{G}$ of a Sylow 5-subgroup $\overline{P}$ of $\overline{G}$ has order 40, and is a maximal subgroup of $\overline{G}$.

SECTION B

4 Define the group $SL_n(q)$, the special linear group over the field $GF(q)$ of q elements, and obtain a formula for its order.

Show that $SL_n(q)$ is generated by its transvections if either $n \geqslant 3$, or if $n = 2$ and $q > 3$.

Use Iwasawa's Lemma (which should be stated but need not be proved) to show that the projective special linear group $PSL_n(q)$ is simple, if either $n \geqslant 3$, or if $n = 2$ and $q > 3$.

Show that $PSL_2(4)$ and $PSL_2(5)$ are both isomorphic to A_5.

Show that $PSL_3(4)$ and $PSL_4(2)$ have the same order but are not isomorphic. (Consider the conjugacy classes of elements of order 2; they are determined by their Jordan Normal Forms.)

5 Define the symplectic group $Sp_{2m}(q)$ over the field $GF(q)$ of q elements, and determine its order. What is the order of $Sp_4(2)$?

The symmetric group S_{2m} acts naturally on a vector space $V_{2m}(2)$ over the field $GF(2)$. Show that there is an invariant subspace of codimension 1, on which the usual dot product gives a symplectic form with a 1-dimensional radical.

Deduce that S_{2m+2} is isomorphic to a subgroup of $Sp_{2m}(2)$ for $m \geqslant 2$, and that $S_6 \simeq Sp_4(2)$.

6 Write an essay on the O'Nan – Scott Theorem.

Illustrate by describing (without proof) the maximal subgroups of S_n for $n = 16$ and $n = 60$.

END OF PAPER