UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2009 9:00 am to 12:00 pm

PAPER 19

COMPLEX MANIFOLDS

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define an *almost complex structure* J on a smooth manifold M and *differential (p,q)-forms* with respect to J. An almost complex structure J is called *integrable* (or *torsion-free*) if $[JX,JY]-[X,Y]-J[X,JY]-J[JX,Y]=0$, for each pair of vector fields X,Y on M. State at least two other, different ways to equivalently define the integrable property of J. If M is a complex manifold, explain what is meant by the almost complex structure J_M induced by the holomorphic atlas, showing that J_M is integrable.

Let $Y\subset M$ be an embedded smooth real submanifold. Prove that if for each $y\in Y$ the tangent space T_yY satisfies $J_M(T_yY)\subseteq T_yY$ then J_M induces, by restriction to TY, an integrable complex structure on Y.

[*Any form of the Inverse Function Theorem and the Implicit Function Theorem may be used without proof if accurately stated.*]

Give a definition of *non-singular analytic subvariety* of a complex manifold. Let $p(z)=z^TQz$, $z\in\mathbb{C}^4$ be a homogeneous quadratic polynomial defined by a symmetric invertible matrix $Q\in GL(4,\mathbb{C})$. Show that $S=\{z\in\mathbb{C}P^3 : p(z)=0\}$ is a non-singular analytic subvariety of $\mathbb{C}P^3$. Show also that S contains two distinct projective lines passing through each point and deduce that S is biholomorphic to $\mathbb{C}P^1\times\mathbb{C}P^1$.

[*You may assume that every $A\in GL(n+1,\mathbb{C})$ induces a biholomorphic map of $\mathbb{C}P^n$ onto itself.*]

2 Define the differential operators ∂, $\bar\partial$, d^c. Verify that for each real differential form η on a complex manifold the differential form $i\partial\bar\partial\eta$ is also real. Define the *Dolbeault cohomology* $H^{p,q}(X)$ of a complex manifold X. Prove that if X is compact and connected then $H^{0,0}(X)\cong\mathbb{C}$.

[*Basic properties of holomorphic functions on domains in $\mathbb{C}$ may be used without proof.*]

Define what is meant by a *holomorphic p-form* on a complex manifold X. Let n denote the complex dimension of X, suppose that X is compact and let $\alpha\not\equiv 0$ be a holomorphic n-form on X. By considering the integral $\int_X \alpha\wedge\bar\alpha$, or otherwise, show that α cannot be d-exact. Show further that if $n=2$, i.e. if X is a compact complex surface, then every holomorphic p-form on X is d-closed and never d-exact, unless identically zero. [It is *not* known whether X admits a Kähler metric.]

[*Standard results about real differential forms and integration over smooth manifolds may be assumed without proof, provided these are clearly stated.*]

3 Define what is meant by an *irreducible analytic hypersurface* Y in a compact complex manifold X and a *local defining function* of Y at a point $p \in X$. Show that a local defining function is uniquely determined, up to multiplication by holomorphic functions f with $f(p) \neq 0$. [You should accurately state any standard results about local rings that you require, but you are not expected to show the existence of a local defining function at singular points of Y.]

Define the *divisors* on X and explain what is meant by the holomorphic line bundle $[D]$ associated to a divisor D. If H_1 and H_2 are two hyperplanes in $\mathbb{C}P^n$, prove that $[H_1]$ and $[H_2]$ are isomorphic holomorphic bundles.

Define the terms *blow-up* $\sigma : \tilde{X} \to X$ at a point $p \in X$ and *exceptional divisor* E on $\tilde{X}$. State the relation between the canonical bundles K_X and $K_{\tilde{X}}$. Let Y be an irreducible non-singular analytic hypersurface in X, such that $p \in Y$, and let $\tilde{Y}$ denote the closure of $\sigma^{-1}(Y \setminus \{p\})$ in $\tilde{X}$. Suppose that the line bundle $[-Y]$ is isomorphic to K_X. Show that $\tilde{Y}$ is a well-defined analytic hypersurface and that $[-\tilde{Y}]$ will be isomorphic to $K_{\tilde{X}}$ if and only if X is a complex surface.

[*You may assume that the transition functions determine a vector bundle up to an isomorphism and that the holomorphic bundles* $[D_1 + D_2]$ *and* $[D_1] \otimes [D_2]$ *are isomorphic for each pair of divisors* D_1, D_2.]

4 Define the terms *holomorphic line bundle* L over a complex manifold X and *local holomorphic section* of L. If L is endowed with a Hermitian inner product on the fibres, explain what is meant by a *Chern connection* on L. Prove the existence and uniqueness of a Chern connection and determine a formula for the curvature of a Chern connection in terms of local holomorphic sections of L.

Now suppose that h_1 and h_2 are two choices of Hermitian inner product on the fibres of L. For $j = 1, 2$, let A_j denote the Chern connections on L determined by h_j, with $F(A_j)$ its curvature form. Show that $iF(A_1)$ and $iF(A_2)$ represent the same Dolbeault cohomology class in $H^{1,1}(X)$.

Let $\hat{L} \to X$ be a holomorphic line bundle, such that $\hat{L}$ admits a nowhere-zero smooth section over X. Show that if $H^{0,1}(X) = 0$ then $\hat{L}$ admits a nowhere-zero holomorphic section over X.

[*Standard properties of connections on vector bundles over smooth manifolds may be used without proof, provided these are accurately stated.*]

5 Let X be a Hermitian manifold. Define the *fundamental form* ω of the Hermitian metric, showing that ω is a real $(1,1)$-form. Define the complex *Hodge $*$-operator*, the *Laplacian* $\Delta = \Delta_d$ and the *complex Laplacian* $\Delta_{\bar{\partial}}$ on X. State the Hodge theorem for the space of (p,q)-forms.

Now suppose that X is compact and Kähler. Show that if α is a (p,q)-form on X ($p > 0$, $q > 0$), $d\alpha = 0$ and α is L^2-orthogonal to the space of harmonic (p,q)-forms then $\alpha = \partial\bar{\partial}\beta$.

Show that if $\dim_{\mathbb{C}} X > 1$ then the restriction of the Lefschetz operator $L(\alpha) = \alpha \wedge \omega$ to 1-forms on X is injective. Show that L commutes with Δ and deduce that the Betti numbers satisfy $b^3(X) \geqslant b^1(X)$.

[*You may assume that on a compact Hermitian manifold* $\bar{\partial}^* = - * \partial *$ *is the formal* L^2 *adjoint of* $\bar{\partial}$ *and that the space of* d*-harmonic* r*-forms is isomorphic to the de Rham cohomology of degree* r. *The identities* $[\bar{\partial}^*, L] = i\partial$ *and* $\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$ *on Kähler manifolds may be used without proof, provided that you include the relevant definitions.*]

END OF PAPER