UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 3 June, 2009 1:30 pm to 4:30 pm

PAPER 18

CURVES AND ABELIAN VARIETIES

*Attempt no more than **THREE** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

In questions (1)-(5) C is a curve of genus $g \geqslant 2$ over an algebraically closed field k.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the Picard functor of degree d line bundles on C, and show that it is representable by a k-variety $Pic_C^d = Jac_C^d$. Show that Jac_C^d is smooth and proper.

2 Define the theta divisor Θ in $X = Jac_C^{g-1}$. For a fixed divisor class D of degree g on C, define $\phi_D : C \to X$ by $\phi_D(P) = [D - P]$. Show that $\phi_D(C)$ is contained in Θ if and only if $\dim H^0(C, \mathcal{O}(D)) \geqslant 2$, and that $\phi_D(C) \cap \Theta = D$ otherwise.

Show that $\dim H^0(X, \mathcal{O}(\Theta)) = 1$.

3 Put $A = Jac_C^0$. Fix a number N prime to *char* k. Show that the N-torsion $A[N]$ is isomorphic to $(Z/NZ)^{2g}$ [you may assume the theorem of the cube]. Define the Weil pairing e_N on $A[N]$, and prove that it is non-degenerate.

4 Consider the abelian sum map $\alpha : C^{3g-3} \to X$.

(i) Show that $\alpha^*\Theta$ is linearly equivalent to $2\sum_1^{3g-3} pr_i^* K_C - \sum_{i<j} \Delta_{ij}$, where K_C is the canonical class and Δ_{ij} is an appropriate diagonal.

(ii) Suppose that $P \in A[N]$. Show that if (σ_i^P) is a basis of $H^0(C, \mathcal{O}(2K_C + P))$ and (σ_i) is a basis of $H^0(C, \mathcal{O}(2K_C))$, then there is a rational function $f_P \in k(X)$ such that $(f_P) = N\Theta_P - N\Theta$, where Θ_P is the translate $t_P^*\Theta = \Theta + P$, whose pull back to C^{3g-3} is given by

$$f_P(z_1, ..., z_{3g-3}) = (\det(\sigma_i^P(z_j))/\det(\sigma_i(z_j)))^N.$$

5 Show that the fibres of the abelian sum map $\alpha : C^{(d)} \to Jac_C^d$ are smooth as schemes.

6 In this question, we work over the complex numbers.

Suppose that C is a compact Riemann surface of genus $g \geqslant 1$. Show that $H_1(C, Z)$ embeds in $H^0(C, \Omega_C^1)^\vee$ as a lattice, and that, having fixed a base point $P_0 \in C$, two points in $C^{(d)}$ have the same image in $H^0(C, \Omega_C^1)^\vee / H_1(C, Z)$ under the map $\sum Q_i \mapsto (\omega \mapsto \sum \int_{P_0}^{Q_i} \omega))$ if and only if they are linearly equivalent as divisors on C.

END OF PAPER