UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 29 May, 2009 9:00 am to 12:00 pm

PAPER 17

DIFFERENTIAL GEOMETRY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let M be a smooth manifold. Define what it means for E to be a smooth vector bundle over M, and what it means to have a smooth metric on E. Show that any vector bundle admits a smooth metric.

(b) Now suppose we have a subbundle $F \subset E$ (i.e. F is another smooth vector bundle that is a submanifold of E and F_p is a linear subspace of E_p for each $p \in M$). Define

$$E/F = \bigsqcup_{p \in M} E_p/F_p$$

and show how E/F can be made into a smooth vector bundle.

(c) Finally show that if F is a subbundle of E then the direct sum bundle $F \oplus (E/F)$ is isomorphic as a vector bundle to E.

[*Hint: start by picking a metric on E. You may assume the direct sum of bundles exists*].

2

(a) Define the space $\Omega^p(M)$ of smooth p-forms on a smooth manifold M. Also define the exterior derivative map $d\colon \Omega^p(M) \to \Omega^{p+1}(M)$ taking care to ensure that what you write is well-defined, and prove that $d^2 = 0$. Explain how the identity $d^2 = 0$ is used to define the p-th de-Rham cohomology group $H^p_{dR}(M)$, and show that $H^p_{dR}(M) = 0$ for $p > \dim M$.

(b) Suppose now that M is the disjoint union of two smooth manifolds U and V. Prove there is an isomorphism

$$H^p_{dR}(M) \simeq H^p_{dR}(U) \oplus H^p_{dR}(V) \qquad (*)$$

for all $p \geqslant 0$.

(c) Now suppose we drop the assumption that U and V are disjoint. Give a proof or counterexample with justification of the isomorphism (*) (i) in the case $p = 0$ and (ii) in the case $p = 1$.

3

(a) Let G be a Lie group, and for each $g \in G$ define $L_g\colon G \to G$ by $L_g(h) = gh$. Define what it means for a vector field X to be left invariant. If $\mathfrak{g}$ is the tangent space to G at the identity, show how $\mathfrak{g}$ is isomorphic to the space of left-invariant vector fields on G and show how this enables you to make $\mathfrak{g}$ into a Lie-algebra.

(b) We say a differential form ω is left invariant if $L_g^*\omega = \omega$ for all $g \in G$. Show that if ω is a left-invariant one-form and X is a left invariant vector field then $\omega(X)$ is constant. Using this or otherwise show that if Y is another left-invariant vector field then

$$d\omega(X, Y) = -\omega([X, Y]).$$

[*Standard identities may be used without proof as long as they are stated clearly.*]

(c) Finally suppose that $\mathfrak{g}$ is abelian (i.e. the Lie bracket $[\psi, \eta]$ vanishes for all ψ, η in $\mathfrak{g}$). Show that if ω is a left-invariant one-form then $d\omega = 0$.

4

(a) Let M be a smooth manifold and g be a Riemannian metric on M. Given a connection ∇ on M define

(i) what it means for ∇ to be compatible with g and

(ii) what it means for ∇ to be symmetric and

(iii) the Levi-Civita connection on M.

[*You are not expected to show the Levi-Civita connection exists.*]

Describe the Levi-Civita connection when $M = \mathbb{R}^n$ and g is the usual Euclidean metric.

(b) Suppose now that ∇ is the Levi-Civita connection on M. Define the curvature tensor $R(X,Y) \in \mathrm{End}(TM)$ for vector fields X, Y and prove the identity

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

(c) Now suppose that $M \subset \mathbb{R}^n$ is an embedded oriented hypersurface, with outward normal vector $\underline{n}$. If X, Y are vector fields on M define

$$\nabla_X(Y) = \tilde{\nabla}_{\tilde{X}}(\tilde{Y}) - < \tilde{\nabla}_{\tilde{X}}(\tilde{Y}), \underline{n} > \underline{n}$$

where $\tilde{X}$ and $\tilde{Y}$ are extensions of X and Y to $\mathbb{R}^n$ and $\tilde{\nabla}$ is the Levi-Civita connection on $\mathbb{R}^n$ (with respect the usual Euclidean metric). Show that ∇ is the Levi-Civita connection on M with respect to the induced metric. [*You may assume the expression for $\nabla_X(Y)$ does not depend on the choice of extensions $\tilde{X}$ and $\tilde{Y}$.*]

5

(a) Let $\pi\colon E \to M$ be a smooth vector bundle on a smooth manifold M. Define what it means for ∇ to be a connection on E, and prove that any smooth vector bundle admits a connection.

(b) Now suppose we have a connection ∇ on E and a local frame $e_1, \ldots, e_r$ of $E|_U$ over some open set $U \subset M$. Define the connection matrix θ and the curvature matrix Θ with respect to this frame and show that

$$d\Theta = \Theta \wedge \theta - \theta \wedge \Theta.$$

(c) If Θ has entries $\Theta = (\Theta_{ij})_{i,j=1}^r$ let

$$\alpha = \sum_{i=1}^{r} \Theta_{ii}.$$

Prove that α does not depend on the choice of frame, and also that α is a closed 2-form.

END OF PAPER