UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2009 9:00 am to 12:00 pm

PAPER 16

ALGEBRAIC TOPOLOGY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Compute the homology groups of the following spaces.

1. Take a solid equilateral triangle, orient the edges to make the boundary an oriented circle, and then form the quotient space which identifies the edges according to the orienting arrows.
2. Attach a Möbius band to the real projective plane by gluing the boundary circle of the band to the circle $\mathbb{RP}^1 \subset \mathbb{RP}^2$ by a map of degree n.
3. Take $S^n \times [0,1]$ and divide by the quotient relation $(x,1) \sim (-x,0)$, where $x \mapsto -x$ denotes the antipodal map.

Can any of these spaces be homotopy equivalent to closed manifolds?

2 Compute the cohomology ring of complex projective space, justifying your argument.

Let U, V and W be complex vector spaces and $\phi : U \otimes V \to W$ a complex-linear map which is injective on each factor $U \otimes v$ and $u \otimes V$ separately, for $u \in U$ and $v \in V$. Prove that the image of ϕ has rank at least $dim_{\mathbb{C}}(U) + dim_{\mathbb{C}}(V) - 1$.

3 Let P denote the real projective plane. Compute the cohomology rings of P and of the product $P \times P$ with $\mathbb{Z}_2$-coefficients.

Suppose $\Sigma \subset P \times P$ is a smooth submanifold which is disjoint from the diagonal $\Delta \subset P \times P$. Can Σ be the graph of a smooth function? Justify your answer.

4 Carefully stating any theorems you invoke, give examples with justification of:

1. A simply-connected manifold on which every smooth function has at least 4 critical points;
2. A simply-connected space on which no finite group acts freely;
3. A vector bundle which has no nowhere-zero section;
4. A map which acts trivially on homology in all degrees > 0 but is not homotopic to a constant.

5 Let M and N be closed oriented manifolds of dimension m and n respectively. For each of the following statements provide a proof or counterexample with justification.

1. If $n = m$ and if M and N are simply connected with torsion-free cohomology and equal Betti numbers, M and N are homotopy equivalent.
2. If M and N are simply-connected and $m > n$, there is a fibre bundle $F \to M \to N$ for some F.
3. If $f : S^m \to M$ is a map of degree 3, then M is not a product $X^k \times Y^{m-k}$ of lower-dimensional manifolds (unless $k = 0$).
4. If $m = 4$ the Euler characteristic of M could be any integer value.

6 The Stiefel manifold $V_k(\mathbb{C}^n)$ is the space of ordered orthonormal k-tuples of vectors in $\mathbb{C}^n$. Show that there is an orientable vector bundle $E \to V_k(\mathbb{C}^n)$ for which the sphere bundle $S(E) \simeq V_{k+1}(\mathbb{C}^n)$. Using the Gysin sequence, prove by induction on k that

$$H^*(V_k(\mathbb{C}^n)) \;\cong\; \Lambda_{\mathbb{Z}}(a_{2n-1}, a_{2n-3}, \ldots, a_{2n-2k+1})$$

where the right hand side denotes the algebra freely generated by elements of the indicated degrees subject only to the constraints imposed by graded commutativity. Deduce that the Betti numbers $b_j(U(n))$ of the unitary group $U(n)$ satisfy

$$\sum_{j \geqslant 0} b_j(U(n)) t^j \;=\; \prod_{j=0}^{n-1} (1 + t^{2j+1}).$$

END OF PAPER