UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2009 9:00 to 11:00 am

PAPER 15

PERCOLATION AND COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\overrightarrow{\Lambda}$ be a connected and locally finite infinite oriented multi-graph, and let x be any site of $\overrightarrow{\Lambda}$.

(i) Show that

$$p_H^b(\overrightarrow{\Lambda};x) \leqslant p_H^s(\overrightarrow{\Lambda};x) \quad \text{and} \quad p_T^b(\overrightarrow{\Lambda};x) \leqslant p_T^s(\overrightarrow{\Lambda};x).$$

(ii) Suppose that $\overrightarrow{\Lambda}$ has maximal in-degree $\Delta_{\text{in}} < \infty$. Bound $p_H^s(\overrightarrow{\Lambda};x)$ away from 1 by a function of $p_H^b(\overrightarrow{\Lambda};x)$ and Δ_{in}.

(iii) Suppose $p_H^b(\overrightarrow{\Lambda};x) < 1/10$. Does it follow that $p_H^s(\overrightarrow{\Lambda};x) < 1$?

2 Consider bond percolation on $\mathbb{Z}^2$ with probability p.

(i) Show that if $p < 1/2$ then there is a constant $a = a(p) > 0$ such that

$$\mathbb{P}_p\big(|C_0| \geqslant n\big) \leqslant \exp(-an)$$

for all $n \geqslant 2$.

(ii) Deduce from the result in (i) that if $p > 1/2$ then there is a constant $b = b(p) > 0$ such that

$$\mathbb{P}_p\big(n \leqslant |C_0| < \infty\big) \leqslant \exp(-b\sqrt{n})$$

for all $n \geqslant 1$.

[The results you use should be stated precisely.]

3 (i) Prove a 0-1 law for translation-invariant events in a translation-invariant site percolation measure.

(ii) Let Λ be a connected and locally finite infinite unoriented graph, which is amenable and of finite type. Let $\mathbb{P}_{\mathbf{p}}$ be a translation-invariant independent site percolation measure on Λ. Write I_k for the event that there are precisely k infinite open clusters. Sketch a proof of the result that $\mathbb{P}_{\mathbf{p}}(I_0) = 1$ or $\mathbb{P}_{\mathbf{p}}(I_1) = 1$. Is this true if $\mathbb{P}_{\mathbf{p}}$ is a k-independent translation-invariant site percolation measure?

(iii) Deduce from the result in (ii) that for bond percolation on $\mathbb{Z}^2$ we have $\theta(1/2) = 0$.

[The terms you use should be defined precisely.]

4 Replace every second lattice point of the honeycomb lattice by a triangle to obtain a cubic plane lattice Λ whose faces are congruent regular triangles and congruent convex enneagons (9-gons) with equal sides (thus every triangle is adjacent to three enneagons, and every enneagon adjacent to three triangles and six enneagons).

(i) Sketch a proof of the relation $p_c^b(\Lambda) + p_c^b(\Lambda^*) = 1$, where Λ^* is the dual of Λ.

(ii) Show that $p_c^b(\Lambda) = 1/\sqrt{2}$.

END OF PAPER