

MATHEMATICAL TRIPOS      Part III

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Thursday, 4 June, 2009    9:00 to 11:00 am

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PAPER 15

PERCOLATION AND COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

**STATIONERY REQUIREMENTS**

*Cover sheet  
Treasury Tag  
Script paper*

**SPECIAL REQUIREMENTS**

*None*

**You may not start to read the questions  
printed on the subsequent pages until  
instructed to do so by the Invigilator.**

**1** Let  $\vec{\Lambda}$  be a connected and locally finite infinite oriented multi-graph, and let  $x$  be any site of  $\vec{\Lambda}$ .

(i) Show that

$$p_H^b(\vec{\Lambda}; x) \leq p_H^s(\vec{\Lambda}; x) \quad \text{and} \quad p_T^b(\vec{\Lambda}; x) \leq p_T^s(\vec{\Lambda}; x).$$

(ii) Suppose that  $\vec{\Lambda}$  has maximal in-degree  $\Delta_{\text{in}} < \infty$ . Bound  $p_H^s(\vec{\Lambda}; x)$  away from 1 by a function of  $p_H^b(\vec{\Lambda}; x)$  and  $\Delta_{\text{in}}$ .

(iii) Suppose  $p_H^b(\vec{\Lambda}; x) < 1/10$ . Does it follow that  $p_H^s(\vec{\Lambda}; x) < 1$ ?

**2** Consider bond percolation on  $\mathbb{Z}^2$  with probability  $p$ .

(i) Show that if  $p < 1/2$  then there is a constant  $a = a(p) > 0$  such that

$$\mathbb{P}_p(|C_0| \geq n) \leq \exp(-an)$$

for all  $n \geq 2$ .

(ii) Deduce from the result in (i) that if  $p > 1/2$  then there is a constant  $b = b(p) > 0$  such that

$$\mathbb{P}_p(n \leq |C_0| < \infty) \leq \exp(-b\sqrt{n})$$

for all  $n \geq 1$ .

[The results you use should be stated precisely.]

**3** (i) Prove a 0-1 law for translation-invariant events in a translation-invariant site percolation measure.

(ii) Let  $\Lambda$  be a connected and locally finite infinite unoriented graph, which is amenable and of finite type. Let  $\mathbb{P}_{\mathbf{p}}$  be a translation-invariant independent site percolation measure on  $\Lambda$ . Write  $I_k$  for the event that there are precisely  $k$  infinite open clusters. Sketch a proof of the result that  $\mathbb{P}_{\mathbf{p}}(I_0) = 1$  or  $\mathbb{P}_{\mathbf{p}}(I_1) = 1$ . Is this true if  $\mathbb{P}_{\mathbf{p}}$  is a  $k$ -independent translation-invariant site percolation measure?

(iii) Deduce from the result in (ii) that for bond percolation on  $\mathbb{Z}^2$  we have  $\theta(1/2) = 0$ .

[The terms you use should be defined precisely.]

4 Replace every second lattice point of the honeycomb lattice by a triangle to obtain a cubic plane lattice  $\Lambda$  whose faces are congruent regular triangles and congruent convex enneagons (9-gons) with equal sides (thus every triangle is adjacent to three enneagons, and every enneagon adjacent to three triangles and six enneagons).

(i) Sketch a proof of the relation  $p_c^b(\Lambda) + p_c^b(\Lambda^*) = 1$ , where  $\Lambda^*$  is the dual of  $\Lambda$ .

(ii) Show that  $p_c^b(\Lambda) = 1/\sqrt{2}$ .

**END OF PAPER**