UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2009 1:30 pm to 3:30 pm

PAPER 14

COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State and prove the Local LYM inequality. State the LYM inequality, and give two proofs: one using Local LYM and one using maximal chains.

State Sperner's lemma on antichains, and explain why it follows from the LYM inequality. Which antichains in $\mathcal{P}([n])$ have size exactly $\binom{n}{\lfloor n/2 \rfloor}$?

A set system $\mathcal{A} \subset \mathcal{P}([n])$ has the property that, for any distinct $i, j \in [n]$, there is a member of $\mathcal{A}$ that contains i but not j. Show that $n \leqslant \binom{m}{\lfloor m/2 \rfloor}$, where $m = |\mathcal{A}|$.

2 Let $1 \leqslant r < n/2$. State the Erdős-Ko-Rado theorem concerning intersecting families in $[n]^{(r)}$.

Explain why the following statement (*) immediately implies the Erdős-Ko-Rado theorem:

(*) If $\mathcal{A} \subset [n]^{(r)}$ is intersecting, and $\mathcal{C}$ is the initial segment of the lexicographic ordering on $[n]^{(r)}$ with $|\mathcal{C}| = |\mathcal{A}|$, then $\mathcal{C}$ is also intersecting.

Use (U, V)-compressions to give a direct proof of (*).

3 State and prove the vertex-isoperimetric inequality in the discrete cube (Harper's theorem).

What does it mean for a sequence of graphs to form a *Lévy family*? Prove that the sequence of discrete cubes $(Q_n)_{n=1}^{\infty}$ forms a Lévy family.

[*Estimates on binomial coefficients may be quoted without proof, provided that they are precisely stated.*]

For a fixed positive integer d, does the sequence of grids $([n]^d)_{n=1}^{\infty}$ form a Lévy family? Justify your answer.

4 State and prove the Frankl-Wilson theorem (on modular intersections).

Let $A \subset \mathcal{P}([n])$ be a family of odd-sized sets such that $|x \cap y|$ is even for all distinct $x, y \in A$. Prove that $|A| \leqslant n$.

Now let n be even, and let $A \subset \mathcal{P}([n])$ be a family of even-sized sets such that $|x \cap y|$ is even for all distinct $x, y \in A$. Give an example with $|A| = 2^{n/2}$. Prove that in fact any such A must satisfy $|A| \leqslant 2^{n/2}$

[*Hint. If $|A| > 2^{n/2}$, what do we know about the dimension of the linear span of (the characteristic vectors of) the points of A?*]

[*Standard facts about linear independence and linear maps may be assumed, provided that they are clearly stated.*]

END OF PAPER