

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2009 1:30 pm to 3:30 pm

PAPER 14

COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight

STATIONERY REQUIREMENTS

*Cover sheet
Treasury Tag
Script paper*

SPECIAL REQUIREMENTS

None

<p>You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.</p>

1 State and prove the Local LYM inequality. State the LYM inequality, and give two proofs: one using Local LYM and one using maximal chains.

State Sperner's lemma on antichains, and explain why it follows from the LYM inequality. Which antichains in $\mathcal{P}([n])$ have size exactly $\binom{n}{\lfloor n/2 \rfloor}$?

A set system $\mathcal{A} \subset \mathcal{P}([n])$ has the property that, for any distinct $i, j \in [n]$, there is a member of \mathcal{A} that contains i but not j . Show that $n \leq \binom{m}{\lfloor m/2 \rfloor}$, where $m = |\mathcal{A}|$.

2 Let $1 \leq r < n/2$. State the Erdős-Ko-Rado theorem concerning intersecting families in $[n]^{(r)}$.

Explain why the following statement (*) immediately implies the Erdős-Ko-Rado theorem:

(*) If $\mathcal{A} \subset [n]^{(r)}$ is intersecting, and \mathcal{C} is the initial segment of the lexicographic ordering on $[n]^{(r)}$ with $|\mathcal{C}| = |\mathcal{A}|$, then \mathcal{C} is also intersecting.

Use (U, V) -compressions to give a direct proof of (*).

3 State and prove the vertex-isoperimetric inequality in the discrete cube (Harper's theorem).

What does it mean for a sequence of graphs to form a *Lévy family*? Prove that the sequence of discrete cubes $(Q_n)_{n=1}^\infty$ forms a Lévy family.

[*Estimates on binomial coefficients may be quoted without proof, provided that they are precisely stated.*]

For a fixed positive integer d , does the sequence of grids $([n]^d)_{n=1}^\infty$ form a Lévy family? Justify your answer.

4 State and prove the Frankl-Wilson theorem (on modular intersections).

Let $A \subset \mathcal{P}([n])$ be a family of odd-sized sets such that $|x \cap y|$ is even for all distinct $x, y \in A$. Prove that $|A| \leq n$.

Now let n be even, and let $A \subset \mathcal{P}([n])$ be a family of even-sized sets such that $|x \cap y|$ is even for all distinct $x, y \in A$. Give an example with $|A| = 2^{n/2}$. Prove that in fact any such A must satisfy $|A| \leq 2^{n/2}$.

[*Hint. If $|A| > 2^{n/2}$, what do we know about the dimension of the linear span of (the characteristic vectors of) the points of A ?*]

[*Standard facts about linear independence and linear maps may be assumed, provided that they are clearly stated.*]

END OF PAPER