UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2009 1:30 pm to 4:30 pm

PAPER 13

ADDITIVE COMBINATORICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

Rough notation as introduced in the course may be used without comment, as may results from Ruzsa calculus.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $N > 1$ be a prime. What is the Fourier transform of a function $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$? State and prove the basic properties of the Fourier transform and use them to prove the following two assertions.

(i) If $A \subseteq \mathbb{Z}/N\mathbb{Z}$ is a set of size αN then $A + A - A - A$ contains a Bohr set of dimension at most $4/\alpha^2$ and width at least $1/10$;

(ii) If $A \subseteq \mathbb{Z}/N\mathbb{Z}$ is a set of size αN with fewer than $\alpha^3 N^2/2$ three-term arithmetic progressions (including the trivial ones) then the balanced function of A has a Fourier coefficient of magnitude at least $c\alpha^C$. [*You may assume the generalized von Neumann theorem if you need it.*]

2 Prove that there is a positive integer n such that $n^2\sqrt{2}$ is within 0.00000001 of an integer. [*The existence of smooth cutoff functions with specified properties may be assumed without proof. Any other results you use should be carefully stated and proved.*]

3 What is meant by a *Freiman isomorphism*? Let A be a finite set of integers. Define the *doubling constant* $\sigma[A]$. If $\sigma[A] \leqslant K$, show that for any prime $p > CK^C|A|$ there is a set $A' \subseteq A$ with $|A'| \geqslant |A|/2$ which is Freiman 2-isomorphic to a subset of $\mathbb{Z}/p\mathbb{Z}$.

Suppose now that $A \subseteq \mathbb{F}_2^\infty$ and that $\sigma[A] \leqslant K$. Show that A is Freiman 2-isomorphic to a subset of $\mathbb{F}_2^m$, where $2^m \leqslant CK^C|A|$. [*Hint: consider a minimal m for which A has a model in $\mathbb{F}_2^m$.*]

4 Let p be a prime and suppose that $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$ are sets with $p^\alpha \leqslant |A| \leqslant p^{1-\alpha}$ and $|B| \geqslant p^\beta$. Show that there is some $b \in B$ for which $|A + b \cdot A| \geqslant |A|^{1+c_{\alpha,\beta}}$, where $c_{\alpha,\beta} > 0$.

END OF PAPER