UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2009 1:30 pm to 4:30 pm

PAPER 12

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let Ω be a bounded domain in $\mathbf{R}^n$.

(i) Let $F = F(x, z, p) : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ be a smooth function. Derive the *Euler-Lagrange equation* satisfied by a critical point $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of the functional

$$\mathcal{F}(u) = \int_\Omega F(x, u, Du).$$

(ii) Let L be the 2nd order linear differential operator defined by

$$L\,u = a_{ij} D_i D_j\, u + b_j D_j\, u + c\, u$$

where a_{ij}, b_j, c are bounded functions on Ω, $c \leqslant 0$ in Ω, and $a_{ij}(x)\zeta_i\zeta_j \geqslant \lambda|\zeta|^2$ for some constant $\lambda > 0$ and all $x \in \Omega$, $\zeta \in \mathbf{R}^n$. State and prove the *weak maximum principle* for L concerning the maximum and minimum values of a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Lu = 0$ in Ω.

(iii) Give an example to show that the weak maximum principle in (ii) need not hold if the hypothesis $c \leqslant 0$ is dropped.

(iv) Suppose now that the function F in (i) is independent of the z variable and is uniformly convex in the p variables. Given $g : \partial\Omega \to \mathbf{R}$, prove that there is at most one critical point $u \in C^2(\overline{\Omega})$ of the functional $\mathcal{F}$ in (i) such that $u = g$ on $\partial\Omega$.

(v) In (iv), can the same conclusion be made if the boundedness hypothesis on Ω is dropped? Justify your answer.

2 Let Ω be a bounded domain in $\mathbf{R}^n$. Consider the divergence form operator $Lu \equiv D_i\left(a_{ij}D_ju\right) + qu$, where $q \in L^\infty(\Omega)$ and a_{ij} are measurable functions satisfying $\lambda|\zeta|^2 \leqslant a_{ij}(x)\zeta^i\zeta^j \leqslant \Lambda|\zeta|^2$ for some constants $\lambda, \Lambda > 0$ and all $x \in \Omega$, $\zeta \in \mathbf{R}^n$. Suppose that $q \leqslant 0$. Let $\psi \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$ be given. Consider the Dirichlet problem

$$Lu = f \text{ in } \Omega \quad \text{and} \quad u = \psi \text{ on } \partial\Omega. \qquad (\star)$$

(i) Define what it means for a function $u \in W^{1,2}(\Omega)$ to be a *weak solution* of $(\star)$.

(ii) Show that $(\star)$ has a unique weak solution $u \in W^{1,2}(\Omega)$.

[If you use a maximum principle, you must prove it.]

(iii) Now suppose additionally that $a_{ij} = a_{ji}$ for $1 \leqslant i, j \leqslant n$. Find an appropriate functional $\mathcal{F}$ of which the Euler-Lagrange equation is $Lu = f$. Use $\mathcal{F}$ and the direct method of the calculus of variations to establish solvability of $(\star)$ in $W^{1,2}(\Omega)$, making clear where in your argument the hypothesis $q \leqslant 0$ is used.

(iv) Give an example of a uniformly elliptic operator L of the form above (with bounded coefficients a_{ij}, q of your choosing) and functions f, ψ to show that $(\star)$ need not have a weak solution in $W^{1,2}(\Omega)$ if we drop the hypothesis $q \leqslant 0$.

[In any part of the problem, standard theorems in linear functional analysis and Sobolev space theory may be used without proof provided they are clearly identified.]

3 Let $u \in W^{1,2}_{\text{loc}}(\Omega) \cap L^2(\Omega)$ be a weak solution of the equation $Lu = f$ where L is the divergence form operator defined by $Lu = D_i(a_{ij} D_j u) + b_j D_j u + cu$. Suppose that $a_{ij}(x)\zeta_i\zeta_j \geqslant \lambda|\zeta|^2$ for some constant $\lambda > 0$ and all $x \in \Omega$, $\zeta \in \mathbf{R}^n$, and that $a_{ij}, b_j, c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$.

(i) Show that for any subdomain $\Omega' \subset\subset \Omega$,

$$\|u\|_{W^{1,2}(\Omega')} \leqslant C\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right)$$

for some constant $C \in (0, \infty)$ depending only on n, λ, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\|b_j\|_{L^\infty(\Omega)}$, $\|c\|_{L^\infty(\Omega)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$.

(ii) For $K > 0$, let

$$\mathcal{S}_K = \{u \in W^{1,2}_{\text{loc}}(\Omega) \cap L^2(\Omega) \,:\, Lu = f \text{ weakly in } \Omega \text{ and } \|u\|_{L^2(\Omega)} \leqslant K\}.$$

If $\{u_k\}$ is a sequence in $\mathcal{S}_K$, prove that there is a subsequence $\{u_{k'}\}$ and a function $u \in \mathcal{S}_K$ such that $u_{k'} \to u$ in $W^{1,2}(\Omega')$ for every subdomain $\Omega' \subset\subset \Omega$.

[Standard theorems in linear functional analysis and Sobolev space theory may be used without proof provided they are clearly identified.]

4 Let Ω be a domain in $\mathbf{R}^n$, $f \in L^2(\Omega)$ and $u \in W^{1,2}(\Omega)$ be a weak solution of $\Delta u = f$ in Ω.

(i) Prove that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ and that for any subdomain $\Omega' \subset\subset \Omega$,

$$\|u\|_{W^{2,2}(\Omega')} \leqslant C \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}\right)$$

where $C \in (0,\infty)$ is a constant depending only on n and $\operatorname{dist}(\Omega', \partial\Omega)$.

(ii) Suppose now that Ω is the half-ball $B_1^+ = B_1(0) \cap \{x^n > 0\}$, and that there exists $\psi \in W^{2,2}(B_1^+)$ such that $u - \psi \in W_0^{1,2}(B_1^+)$. Prove that $u \in W^{2,2}(B_{1/2}^+)$, where $B_{1/2}^+ = B_{1/2}(0) \cap \{x^n > 0\}$, and that

$$\|u\|_{W^{2,2}(B_{1/2}^+)} \leqslant C \left(\|u\|_{W^{1,2}(B_1^+)} + \|f\|_{L^2(B_1^+)} + \|\psi\|_{W^{2,2}(B_1^+)}\right)$$

for some constant $C \in (0,\infty)$ depending only on n.

[You need not prove the "difference quotient lemmas."]

5 (i) State and prove the mean value properties for a C^2 harmonic function on a domain in $\mathbf{R}^n$.

(ii) State and prove the Harnack inequality for a non-negative C^2 harmonic function on a domain in $\mathbf{R}^n$.

(iii) Let $u \in C^2(\mathbf{R}^n)$ be a bounded function. If u is harmonic in $\mathbf{R}^n$, prove that u must be constant.

(iv) Recall that the Bernstein theorem for minimal graphs says that if $1 \leqslant n \leqslant 7$, and if $u \in C^2(\mathbf{R}^n)$ is a solution of the minimal surface equation

$$D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = 0$$

on $\mathbf{R}^n$, then u is an affine function. Use the Bernstein theorem to prove that if $1 \leqslant n \leqslant 7$ and $u \in C^2(\mathbf{R}^n)$ solves

$$D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = \kappa$$

for some constant κ, then u is an affine function.

6 Let $F : \mathbf{R}^n \to \mathbf{R}$ be a C^2 function with bounded second derivatives. Suppose that F is uniformly convex on $\mathbf{R}^n$, and that there exists a number $\alpha \in (0, \infty)$ such that $F(p) \geqslant \alpha |p|^2$ for all $p \in \mathbf{R}^n$. Let Ω be a bounded domain in $\mathbf{R}^n$, and let

$$\mathcal{F}(u) = \int_\Omega F(Du).$$

(i) Show that $\mathcal{F}(u) < \infty$ for every $u \in W^{1,2}(\Omega)$.

(ii) Prove that the functional $\mathcal{F}$ is weakly lower semi-continuous on $W^{1,2}(\Omega)$.

(iii) Let $\psi \in W^{1,2}(\Omega)$ be given, and define $\mathcal{C}_\psi = \{u \in W^{1,2}(\Omega) : u - \psi \in W_0^{1,2}(\Omega)\}$. Prove that there exists a function $u \in \mathcal{C}_\psi$ such that

$$\mathcal{F}(u) = \inf_{v \in \mathcal{C}_\psi} \mathcal{F}(v).$$

(iv) Stating any required additional hypotheses on F, and quoting the relevant intermediate theorems without proof, briefly explain how to obtain interior $C^{1,\beta}$ regularity of u for some $\beta \in (0, 1)$.

[In any part of the problem, standard theorems in linear functional analysis and Sobolev space theory may be used without proof provided they are clearly identified.]

END OF PAPER