UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2009 9:00 am to 12:00 pm

PAPER 11

TOPICS IN COMPLEX ANALYSIS

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain why the series

$$\varepsilon_1(z) = \sum_{n \in \mathbb{Z}\setminus\{0\}} \left(\frac{1}{z-n} + \frac{1}{n} \right) + \frac{1}{z} \qquad \text{and} \qquad \varepsilon_k(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^k} \quad \text{for } k = 2, 3, 4, \ldots$$

define meromorphic functions on the complex plane. Show carefully that $\varepsilon_k(z) = \varphi_k(e^{2\pi i z})$ for some rational function ϕ_k and find the functions φ_1, φ_2 explicitly.

Find the Laurent series for ε_1 on the annulus $\{z : 0 < |z| < 1\}$, expressing the coefficients in terms of the Riemann ζ-function.

2 Let ρ denote the hyperbolic metric on the unit disc $\mathbb{D}$.

For which sequences of points (z_n) in $\mathbb{D}$ is there a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ with zeros at the points z_n and nowhere else? Justify your answer briefly.

Show that there is a bounded holomorphic function $f : \mathbb{D} \to \mathbb{C}$ with zeros at the points z_n and nowhere else if and only if

$$\sum \exp(-\rho(0, z_n)) < \infty \; .$$

Prove that

$$\log \frac{1+t}{1-t} \geqslant 2t \qquad \text{for } t \in [0, 1) \; .$$

By setting $t = e^{-\rho(w, z_n)}$, or otherwise, show that

$$|B(w)| \leqslant \exp\left(-2 \sum e^{-\rho(w, z_n)}\right)$$

for any point $w \in \mathbb{D}$ and for B a Blaschke product with zeros (z_n).

3 State and prove Runge's Theorem.

Show that we can find polynomials P_n for which $(P_n(z))$ converges at each point of $\mathbb{C}$ and

$$P_n(z) \to \begin{cases} +1 & \text{for } \mathrm{Im}(z) > 0; \\ 0 & \text{for } \mathrm{Im}(z) = 0; \\ -1 & \text{for } \mathrm{Im}(z) < 0. \end{cases}$$

Let K be a compact subset of the plane domain Ω. Let K^* be the set of points $w \in \Omega$ for which

$$|f(w)| \leqslant \sup\{|f(z)| : z \in K\}$$

for every holomorphic function $f : \Omega \to \mathbb{C}$. Prove that K^* is the union of K with those components of $\mathbb{P}\setminus K$ that lie entirely within Ω.

4 State and prove the Schwarz – Pick lemma.

Let $\mathcal{F}$ be the set of all holomorphic functions

$$f : \mathbb{D} \to A = \{z \in \mathbb{C} : e^{-1} < |z| < e\}$$

from the unit disc $\mathbb{D}$ into the annulus A with $f(0) = 1$. Explain why there is a function $f \in \mathcal{F}$ for which $|f'(0)|$ is maximal. Find such a function explicitly. Is it unique?

5 Let G be the set of Möbius transformations

$$z \mapsto \frac{uz + v}{\overline{v}z + \overline{u}}$$

where u, v are Gaussian integers with $|u|^2 - |v|^2 = 1$ and $u + v - 1 \in 2\,\mathbb{Z}[i]$. Explain briefly why this is a group of hyperbolic isometries of the unit disc and acts discontinuously on that disc.

Show that the origin is only fixed by the identity transformation in G. Find the Dirichlet region for this group centred on 0 and justify your answer.

END OF PAPER