UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 29 May, 2009 9:00 am to 12:00 pm

PAPER 10

INTRODUCTION TO FUNCTIONAL ANALYSIS

Attempt no more than ***THREE*** *questions, and not more than* ***TWO*** *from either section.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1 Suppose that f is a bounded convex function on the open unit ball U of a real normed space $(E, \|.\|)$, and that $x \in U$.

(a) Show that f is continuous.

(b) Explain briefly why the directional derivative

$$D_y(f)(x) = \lim_{\lambda \searrow 0} \left(\frac{f(x+\lambda y) - f(x)}{\lambda} \right)$$

exists, for each $y \in E$.

(c) Show that $D_y(f)(x)$ is a sublinear function of y.

(d) Show that there exists a linear functional l on E such that $f(x+z) \geqslant f(x)+l(z)$ for all z for which $x+z \in U$.

(e) Show that l is continuous.

(f) Suppose further that $\mathbf{P}$ is a probability measure on the Borel sets of U, and that $\phi(x) = \mathbf{E}(\phi)$ for each $\phi \in E'$ (x is the *barycentre* of $\mathbf{P}$). Show that $\mathbf{E}(f) \geqslant f(x)$ (Jensen's Inequality).

2 What is a *completely regular* Hausdorff topological space?

Suppose that (X, τ) is a completely regular Hausdorff topological space. Let $C_b(X)$ be the space of continuous bounded real-valued functions on X, with the supremum norm, and with dual $C_b(X)'$. Show that the evaluation mapping $\delta : X \to C_b(X)'$ is a homeomorphism, when $C_b(X)'$ is given the weak* topology.

Explain how this is used to define the *Stone-Čech* compactification βX of X. Show that $C(\beta X)$, with the supremum norm, is isometrically isomorphic to $C_b(X)$. Show that if f is a continuous mapping of X into a compact Hausdorff space K then there is a unique continuous extension from βX into K.

[You should state any properties of weak* topologies that you need, but may use them without proof.]

Show that X is open in βX if and only if X is locally compact.

Show that if τ is the discrete topology and $A \subseteq \beta X$ then $\bar{A}$ is open.

3 Let $C(K)$ be the Banach space of continuous real-valued functions on a compact Hausdorff space K.

(a) Show that a positive linear functional ϕ on $C(K)$ is continuous, and that $\|\phi\| = \phi(1)$.

(b) Show that if ϕ is continuous and $\|\phi\| = \phi(1)$ then ϕ is positive.

(c) Show that if ϕ is a continuous linear functional on $C(K)$ then $\phi = \phi^+ - \phi^-$, where ϕ^+ and ϕ^- are positive linear functionals with $\|\phi\| = \|\phi^+\| + \|\phi^-\|$.

(d) Suppose that ϕ is a continuous linear functional on the complex Banach space $C_{\mathbf{C}}(K)$ of continuous complex-valued functions on a compact Hausdorff space K, which satisfies $\phi(1) = 1 = \|\phi\|$. Show that if $g \in C_{\mathbf{C}}(K)$ is real-valued then $\phi(g)$ is real. [Hint: consider $1+itg$, with t real.]

SECTION II

4 Suppose that B is a unital Banach algebra, and that A is a closed unital subalgebra of B. Let

$$\begin{aligned} G(A) &= \{a \in A : a \text{ has an inverse in } A\} \\ G_B(A) &= \{a \in A : a \text{ has an inverse in } B\}. \end{aligned}$$

(a) Show that $G(A) \subseteq G_B(A)$, and give an example to show that the inclusion can be strict.

(b) Show that 1 is an interior point of $G(A)$.

(c) Show that $G(A)$ and $G_B(A)$ are open subsets of A.

(d) Suppose that $a, b \in A$ and that $1 - ab \in G(A)$. Show that $1 - ba \in G(A)$.

(e) Suppose that $a_n \in G(A)$, $a_n \to a$ and that $a \notin G(A)$. Show that $\left\|a_n^{-1}\right\| \to \infty$ as $n \to \infty$. Show that $a \notin G_B(A)$.

(f) Show that $G(A)$ is the union of some of the connected components of $G_B(A)$.

5 (a) State the Gelfand-Mazur theorem.

(b) Suppose that A is a commutative unital Banach algebra. Show that every ideal in A is contained in a maximal proper ideal.

(c) What is a *character* on A? Show that there is a natural bijection from the set Φ_A of characters on A onto the set $\mathcal{M}_A$ of maximal ideals.

(d) Establish how Φ_A is used to determine the spectrum of an element of A.

(e) Let $A = C^1[0,1]$ be the Banach space of continuous functions on $[0,1]$ with continuous derivative (one-sided at 0 and 1), under the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Show that A is a unital commutative Banach algebra, under pointwise multiplication. Determine $\mathcal{M}_A$.

END OF PAPER