UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 3 June, 2009 1:30 pm to 4:30 pm

PAPER 1

DECISION PROBLEMS IN GROUP THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let G be a group and $x_1, \ldots, x_n, g \in G$. Let X be the subgroup of G generated by $x_1, \ldots, x_n$ and K be the normal subgroup of G generated by g. If X is free on $x_1, \ldots, x_n$ and $K \cap X = \{1\}$, prove that the subgroup generated by $x_1 g, \ldots, x_n g$ is free on these generators.

(b) Let G be the group which is finitely presented as a nilpotent class 2 group by $\langle x, y : [x^3, y] = 1 \rangle$. Determine finite presentations for $G/\gamma_2(G)$ and $\gamma_2(G)$. Use them to solve the word problem for G illustrating with an example.

[*You may assume that in any nilpotent class 2 group*

$$[ab, c] = [a, c][b, c] \text{ and } [a, bc] = [a, b][a, c] \quad]$$

2 (a) Let G be the finitely presented group

$$\langle x, y, z : [x, y] = [y, z] = [x, z] = 1, \quad x^2 y^6 = 1, \quad x y^2 z^2 = 1 \rangle.$$

Describe the set of words $x^m y^n z^k$ that are the identity in G and hence solve the word problem for G.

(b) State an algebraic characterization for a finitely generated group to have soluble word problem. Use it and the existence of a finitely presented group with insoluble word problem (both of which you may assume) to prove that there is no algorithm to determine whether or not an arbitrary finitely presented group is simple.

[*You may assume the Messuage Lemma provided you state it clearly.*]

3 Describe what is meant by an *instruction* and an *instantaneous description* (i.e, internal configuration) of a Turing machine T. Given a Turing machine T, give a finitely presented semigroup $\gamma(T)$ with elements h, q such that if w is any word in the alphabet of T, then $hq_1wh = q$ in $\gamma(T)$ iff T halts when given input w, where q_1 is the initial state of T. Prove your result.

4 (a) Let G be a group, $g \in G$ and $n \in \mathbb{Z}_+$. Give a group G_1 containing G as a free factor, an HNN-extension H of G_1 and an element of $h \in H$ with $h^n = g$, proving all your claims.

(b) Let $H = \langle h_1, \ldots, h_m : w_1(\mathbf{h}) = 1, \ldots, w_n(\mathbf{h}) = 1 \rangle$ be a finitely presented group with insoluble word problem. Let F be the free group on $x_1, \ldots, x_m$ and $P = F \times F$. Let L be the subgroup of P generated by

$$\{(x_i, x_i) : i = 1, \ldots, m\} \cup \{(1, w_j(\mathbf{x})) : j = 1, \ldots, n\}$$

and $G = \langle P, t : t^{-1}yt = y \qquad (y \in L) \rangle$.

Give G as a finitely presented group and prove that it has insoluble word problem. (So an HNN-extension of a direct product of two free groups can have insoluble word problem!)

5 A group G is said to be torsion-free if the only element of G of finite order is the identity. Use Britton's Lemma to prove that if H is any HNN-extension of a torsion-free group G, then H is torsion-free. Use this to prove that every countable torsion-free group can be embedded in one in which any two non-identity elements are conjugate.

6 Prove or give a counter example to each of the following.

(i) Any countable group can be embedded in a group in which any two non-identity elements are conjugate.

(ii) If n and m are positive integers greater than 1, then the free group on n generators can be embedded in the free group on m generators iff $n \leqslant m$.

(iii) Every countable group can be embedded in a finitely presented group.

(iv) If finitely presented groups G and H have soluble word problem, then so does $G \times H$.

(v) Any homomorphic image of a finitely presented group is finitely presented.

(vi) If a finitely presented group has soluble word problem, then so does any HNN-extension of it.

END OF PAPER