MATHEMATICAL TRIPOS **Part III**

Thursday 5 June 2008 1.30 to 3.30

PAPER 90

ENVIRONMENTAL FLUID DYNAMICS

You may attempt **ALL** *questions, although high marks can be achieved by good answers to* **THREE** *quesitons.*

Completed answers are preferred to fragments.

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a steady flow of water of depth $h(x)$ and velocity $u(x)$ in the positive x direction along a channel with a rectangular cross-section of width $b(x) = 1 + \beta(x - a)^2$ and bottom elevation $H(x) = -\eta x^2$. [The height of the free surface is therefore $H + h$.] The constants β and η describe the relative strength of the width and depth variations, respectively, and a determines the position of the narrowest point in the channel.

(a) Derive the shallow water specific energy function E and determine the relationship between this and the Froude number F. Define the terms 'subcritical' and 'supercritical' and describe the concept of 'hydraulic control'.

(b) Over what range of x can the hydraulic control be located? Show that if the hydraulic control is located at $x = a/2$ then the depth at this point is

$$h = \frac{\eta}{\beta}\left(1 + \frac{\beta}{4}\,a^2\right) .$$

(c) Suppose the flow along the channel experiences a drag that results in a reduction of hydrostatic head at a rate of γ per unit length along the channel in the vicinity of the hydraulic control. Derive a modified specific energy function to take this head loss into account. In the case of a channel with $\beta = 0$, determine the location of the hydraulic control. Determine also the flow rate Q required to yield a water depth at the control of exactly twice that at $x = 0$.

2 Consider a Boussinesq shallow water flow along the bottom of a channel with a triangular cross-section of width $b(z) = z$, for $z > 0$, where z is directed upwards. The density of the current is $\rho(x,t)$ while the density of the fluid is $\rho_0 = const$. There is negligible drag between the current and the channel, but turbulence within the current leads to an entrainment velocity w_e into the current across the interface.

(a) State what is meant by the terms 'shallow water' and 'Boussinesq', outlining any assumptions. Define a suitable 'reduced gravity' g'.

(b) Assume there is negligible motion in the ambient fluid and derive the shallow water equations for the depth $h(x,t)$, velocity $u(x,t)$ and reduced gravity $g'(x,t)$.

(c) Determine the characteristics of this system and show that the long wave speed is $c = (\frac{1}{2}\,g'h)^{\frac{1}{2}}$. Determine also the equations along the characteristics.

3 Small particles with a volume concentration ϕ are suspended in a fluid. The particles have a settling velocity $V(\phi) = (1 - \phi/\phi_{max})V_s$, where V_s is the settling velocity of an isolated particle and ϕ_{max} is a constant.

(a) Explain the concept of hindered settling. Derive a one-dimensional conservation equation for particles settling in a quiescent fluid and determine the characteristics. State the condition required for a shock to form. Give the jump conditions describing the motion of the shock and show that the shock moves at a speed equal to the average of the characteristics on either side.

(b) For an initial distribution of particles described by $\phi_0 = \frac{1}{3}\,\phi_{max}(1 - (z/h)^2)$, where h is the depth of the layer of quiescent fluid containing particles, determine the time and location at which a shock first forms. What is the initial speed of the shock? Does the shock speed up or slow down after it forms? Justify your answer, but you need not attempt a detailed analysis the behaviour of the flow at later times.

(c) Consider a particle-laden turbulent gravity current of constant volume $M = L_0 h_0$ in a channel of unit width. Here, L_0 and h_0 are the initial length and depth of the current; the initial particle concentration ϕ_0 is uniform within the current. The density of the ambient fluid is ρ_0, the density of the particles is ρ_p, and $\phi_{max} = 1$. Give an expression for the bulk density and the reduced gravity of the current. What condition must be satisfied for the particles to remain well-mixed within the current? Why can the particles still deposit on the bottom of the channel? Derive an integral model for the current assuming sedimentation occurs. Determine the run-out length L_∞ of the current. Show that when $L_\infty \gg L_0$ and $\phi_0 \ll 1$,

$$L_\infty^5 \approx \left(\frac{25F^2M^3g_0'}{V_s^2}\right)\left(\phi_0 + \frac{2}{3}\,\phi_0^2 + O(\phi_0^3)\right).$$

State any additional assumptions you make.

4 Consider a two-dimensional, Boussinesq, time-dependent, turbulent buoyant plume rising from a line source in a stratified environment of density $\rho_0(z)$. The momentum equation, assuming top-hat profiles for all plume properties, may be written as

$$\frac{\partial}{\partial t}(\rho b w) + \frac{\partial}{\partial z}(\rho b w^2) = (\rho_0 - \rho)\, g b\,,$$

where the plume velocity, half-width and density are $w(z,t)$, $b(z,t)$ and $\rho(z,t)$, respectively. Here, g is gravity and z is directed vertically upwards.

(a) Define the entrainment coefficient α and give an expression for the entrainment velocity u_e. Derive equations for conservation of volume and mass, and hence show that conservation of buoyancy may be written in the form

$$\frac{\partial}{\partial t}((\rho_0 - \rho) g b) + \frac{\partial}{\partial z}((\rho_0 - \rho) g b w) = -\rho_0 N^2 b w\,,$$

where $N = \left(-\frac{g}{\rho_0}\frac{d\rho_0}{dz}\right)^{\frac{1}{2}}$ is the buoyancy frequency.

(b) Define the mass flux Q, buoyancy flux F and momentum flux M. Rewrite the mass, momentum and buoyancy equations for the plume in terms of these variables.

(c) Determine the time-dependent power-law solution for a plume in a homogeneous ambient fluid, and hence show that the plume width is given by $b = \frac{1}{2}\alpha z$.

END OF PAPER