MATHEMATICAL TRIPOS Part III

Friday 30 May 2008 1.30 to 4.30

PAPER 88

WAVES IN FLUIDS

Attempt no more than **FOUR** *questions.*

There are **FIVE** *questions in total.*

Completed answers are substantially preferred to fragments.

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) The equation describing aerodynamic sound generation in the presence of an external force distribution is

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \nabla^2 \rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} - \frac{\partial f_i}{\partial x_i} ,$$

where for an inviscid fluid $T_{ij} = \rho u_i u_j + (p - c_0^2 \rho)\delta_{ij}$ is the quadrupole distribution and $f_i(\mathbf{x}, t)$ is the external force per unit volume.

Show that the far-field sound generated by a compact distribution of quadrupoles and forces is

$$\rho'(\mathbf{x}, t) = \frac{x_i x_j \ddot{S}_{ij}(t - |\mathbf{x}|/c_0)}{4\pi |\mathbf{x}|^3 c_0^4} + \frac{x_i \dot{F}_i(t - |\mathbf{x}|/c_0)}{4\pi c_0^3 |\mathbf{x}|^2}$$

where

$$S_{ij}(t) = \int T_{ij}(\mathbf{y}, t)\, d^3y \qquad F_i(t) = \int f_i(\mathbf{y}, t)\, d^3y \, .$$

[You may use the fact that the free-space Green's function in three dimensions is

$$\frac{\delta(t - |\mathbf{x}|/c_0)}{4\pi c_0^2 |\mathbf{x}|} \, . \quad]$$

A small body of characteristic size l is placed in a flow of characteristic speed u. Show how the far-field sound scales with u/c_0.

(b) An unsteady source of strength $q(t)$ located at the point $\mathbf{y} = \mathbf{y}(t)$ produces an acoustic field

$$\rho'(\mathbf{x}, t) = \frac{q(\tau)}{4\pi c_0^2 |1 - M_r| |\mathbf{x} - \mathbf{y}(\tau)|} ,$$

where τ is the emission time and M_r is the Mach number of the source motion in the observer direction at emission.

Consider a point source which rotates with constant angular velocity Ω in a circle of radius a in the plane $z = 0$, so that the coordinates of the source are $(a\cos\Omega t, a\sin\Omega t, 0)$. Show that the acoustic density fluctuation in the far field is

$$\frac{q}{4\pi c_0^2 \Omega |\mathbf{x}|} \frac{\mathrm{d}\Theta}{\mathrm{d}t} ,$$

where

$$\theta - \frac{\pi}{2} + \Theta = \Omega\left(t - \frac{|\mathbf{x}|}{c_0}\right) + \frac{\Omega a r}{c_0 |\mathbf{x}|} \sin\Theta$$

and r, θ, z are the polar coordinates of the observer.

2 (a) Consider the Sommerfeld problem of diffraction of a plane wave by a sharp edge, i.e. solve

$$(\nabla^2 + k_0^2)\phi = 0$$

subject to

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi_i}{\partial y} = 0 \quad \text{on } y = 0, x < 0,$$

where

$$\phi_i = \exp(ik_0 x \cos\theta_0 + ik_0 y \sin\theta_0 + i\omega t)$$

is the incident potential, $\phi(x, y)\exp(i\omega t)$ is the scattered potential and $k_0 = \omega/c_0$.

Find an expression for the far-field scattered potential, comprising diffracted and geometrical optics components. You need not consider the details of the Fresnel regions, and you may quote without proof the result

$$\int_\Gamma f(k)\exp(-ikr\cos\theta - \gamma r|\sin\theta|)dk \sim \left(\frac{2k_0\pi}{r}\right)^{1/2} f(k_0\cos\theta)|\sin\theta|\exp(-ik_0 r + i\pi/4)$$

as $r \to \infty$, where $\gamma = \sqrt{k^2 - k_0^2}$, and Γ is the steepest descent contour, which crosses the real k axis at $k = k_0\cos\theta$ and $k = k_0 \sec\theta$.

(b) Define $\phi(\mathbf{r}|\mathbf{r}_i)$, $i = 1, 2$, to be the potential at $\mathbf{r}$ due to a unit monopole source at $\mathbf{r}_i$, i.e.

$$(\nabla^2 + k_0^2)\phi(\mathbf{r}|\mathbf{r}_i) = \delta(\mathbf{r} - \mathbf{r}_i)\ ,$$

subject to the boundary condition

$$\alpha(\mathbf{r})\phi(\mathbf{r}|\mathbf{r}_i) + \beta(\mathbf{r})\frac{\partial}{\partial n}\phi(\mathbf{r}|\mathbf{r}_i) = 0 \tag{1}$$

on the closed surface S, where $\partial/\partial n$ denotes the normal derivative on S. Show that

$$\phi(\mathbf{r}_1|\mathbf{r}_2) = \phi(\mathbf{r}_2|\mathbf{r}_1)\ .$$

Now suppose that $\psi(\mathbf{r}|\mathbf{r}_1)$ satisfies

$$\frac{1}{f}\nabla.(f\nabla\psi) + k_0^2\psi = \delta(\mathbf{r} - \mathbf{r}_1)\ ,$$

where $f(\mathbf{r})$ is a known function, subject to boundary condition (1). What equation and boundary condition must $\tilde{\psi}(\mathbf{r}|\mathbf{r}_2)$ satisfy in order that

$$\psi(\mathbf{r}_2|\mathbf{r}_1) = \tilde{\psi}(\mathbf{r}_1|\mathbf{r}_2) \quad ?$$

3 (a) Show that Burgers equation

$$\frac{\partial q}{\partial Z} - q\frac{\partial q}{\partial \theta} = \epsilon\frac{\partial^2 q}{\partial \theta^2}$$

can be transformed into the diffusion equation

$$\frac{\partial \psi}{\partial Z} = \epsilon\frac{\partial^2 \psi}{\partial \theta^2}$$

by the transformation $q = 2\epsilon\psi_\theta/\psi$.

Starting from the general solution of the diffusion equation

$$\psi(\theta, Z) = \frac{1}{(4\pi\epsilon Z)^{1/2}}\int_{-\infty}^{\infty}\psi(\theta', 0)\exp\left(-\frac{(\theta'-\theta)^2}{4\epsilon Z}\right)\mathrm{d}\theta'$$

use the initial data

$$\psi(\theta, 0) = \sum_{j=1}^{N}\exp\left(\frac{q_j(\theta-\theta_j)}{2\epsilon}\right) ,$$

where $q_1 < q_2 < .... < q_N$, to derive a solution of Burgers equation. Describe the case $N = 2$, and for general N find the behaviour as $Z \to \infty$.

(b) Consider a linearly polarized, time-harmonic electromagnetic wave with electric field transverse to the z-axis and aligned with the y-axis (so $\mathbf{E} = \hat{\mathbf{y}}E_y$), propagating in an isotropic and inhomogeneous medium with permittivity $\epsilon = \epsilon(z)$ and permeability $\mu = \mu(z)$.

Derive the Helmholtz equation for the y component of the electric field E_y and the z component of the magnetic field H_z.

(c) Now assume that the medium has constant permeability and that the wavenumber $k(z) = (\omega^2\mu\epsilon)^{1/2}$ is given as a function of a known medium profile as

$$k^2(z) = k_0^2 n^2(z) = k_0^2\left(a^2 - \frac{b^2}{(z-z_0)^2}\right) , \tag{2}$$

where a, b and z_0 are arbitrary constant. Consider a TE wave obliquely incident upon the medium:

$$E_y(x, z) = e_y(z)e^{-ik_0\sin\theta_0 x}$$

Derive the solution $e_y(z)$.

[Use the known result that the differential equation

$$\frac{d^2W}{dz^2} + \left(\beta^2 - \frac{4\nu^2-1}{4z^2}\right)W = 0 \tag{3}$$

has solution

$$W(z) = \sqrt{z}Z_\nu(\beta z) ,$$

where $Z_\nu(\beta z)$ is a Bessel function.]

4 Consider a time-harmonic acoustic wave ψ with wavenumber k propagating in a random medium with refractive index

$$n = 1 + \mu W(x, y, z) , \quad \text{with } W = 0 \text{ for } x < 0 , \tag{4}$$

where μ is a constant, and W is normally distributed and stationary in y and z, with $< W >= 0$ and $< W^2 >= 1$.

(a) Under the assumption that propagation is mainly in the forward direction, at a small angle to the horizontal x, so the parabolic equation holds for the reduced wave $E(x, y, z)$, derive an evolution equation for the second moment of the field

$$m_2(x) =< E(x, y_1, z_1)E^*(x, y_2, z_2) >$$

(b) Write the solution for the second moment $m_2(x)$ at an arbitrary value of x, due to a plane wave normally incident on the plane at $x = 0$.

Assuming now that the medium is δ-correlated in the direction of propagation x, i.e.

$$< W(x_1, y_1, z_1)W(x_2, y_2, z_2) >= \delta(x_1 - x_2) < W(y_1, z_1)W(y_2, z_2) > , \tag{5}$$

and isotropic in the (y, z) plane, express this solution in terms of the power spectrum of the medium.

[*You will need to use the following result:*

If

$$F(\nu_\eta, \nu_\zeta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\eta, \zeta) e^{-i(\nu_\eta \eta + \nu_\zeta \zeta)} d\eta d\zeta \tag{6}$$

and the kernel is isotropic, i.e. $f(r, \theta) = f(r)$ *in polar coordinates* (r, θ), *the following applies:*

$$F(\nu_\eta, \nu_\zeta) = F(\nu) = \int_0^{\infty} f(r) J_0(\nu r) r dr , \quad \textit{where } \nu = |(\nu_\eta, \nu_\zeta)| , \tag{7}$$

together with the inverse transform

$$f(\nu) = \int_0^{\infty} F(\nu) J_0(\nu r) \nu d\nu . \tag{8}$$

Here $J_0(z)$ *is the zeroth-order Bessel function, with* $J_0(0) = 1$.]

(c) Derive the Rytov approximation for ψ, to first order, and use it to write an expression for the first moment of the field $< \psi >$

5 Consider a time-harmonic plane wave $\psi_i(\mathbf{r}) = e^{i\mathbf{k_0}\cdot\mathbf{r}}$ incident on a finite, extended inhomogeneity with refractive index $n(\mathbf{r})$, embedded in free space, where $\mathbf{r} = (x, y, z)$.

(a) Find a solution for the scattered field $\psi_s(\mathbf{r})$, using the first order Born approximation . [*Use the following representation of the Green's function* $\frac{e^{ikr}}{r}$ *as a spectrum of plane waves:*

$$\frac{e^{ikr}}{r} = \frac{ik}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{m}e^{ik[p(x-x')+q(y-y')+m(z-z')]}dpdq \tag{9}$$

where

$$m = \begin{array}{lll} (1-p^2-q^2)^{1/2} & \textit{when} & (p^2+q^2) \leqslant 1 \\ i(p^2+q^2-1)^{1/2} & \textit{when} & (p^2+q^2) > 1 \end{array} \tag{10}$$

and $p, q, \in \mathbb{R}$]

(b) Consider now the inverse problem of finding $n(\mathbf{r})$ from the knowledge of the scattered field $\psi_s(\mathbf{r})$ at two values of z either side of the inhomogeneity.

Solve this inverse problem by using the expression obtained in (a) for $\psi_s(\mathbf{r})$ to write the 2D Fourier transform of $V(\mathbf{r}) = -k_0(n(\mathbf{r}) - 1)$ in terms of the 2D Fourier transform of $\psi_s(\mathbf{r})$.

(c) The result obtained in (b) cannot be obtained with $(p^2 + q^2) > 1$. Comment on the physical significance of excluding this case.

Comment on the range of validity of this approximation, as it relates both to the steps needed in the theoretical model to achieve the solution, and to the numerical implementation needed, which will require approximation of the Fourier transforms with discrete, finite sums.

(d) Give at least one reason why this inverse problem is ill-posed, and give an explicit expression for a suitable Tikhonov functional that can be used to regularise this problem.

END OF PAPER