UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 3 June 2008 9.00 to 11.00

PAPER 87

TURBULENCE

*Attempt **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
A data sheet of 3 pages is attached

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Consider homogeneous, isotropic turbulence. Show that, if the two-point correlation $\langle \mathbf{u} \cdot \mathbf{u}' \rangle$ decays sufficiently rapidly with separation $r = |\,\mathbf{r}\,| = |\,\mathbf{x}' - \mathbf{x}\,|$, then the low-wavenumber end of the energy spectrum takes the form

$$E(k) = \frac{Lk^2}{4\,\pi^2} + \frac{Ik^4}{24\,\pi^2} + O(k^6)\,,$$

where k is the wavenumber and

$$L = \int \langle \mathbf{u} \cdot \mathbf{u}' \rangle \, d\mathbf{r}\,, \qquad I = -\int r^2 \langle \mathbf{u} \cdot \mathbf{u}' \rangle \, d\mathbf{r}\,.$$

Express L in terms of the linear momentum in some large volume, V, and use the central limit theorem to argue that, in general, we might expect L to be non-zero. Under what particular conditions might we expect L to be zero?

(ii) The longitudinal triple correlation function, $K(r)$, falls no more slowly than $K_\infty \sim ar^{-4} + br^{-5} + \ldots$, where a and b are constants. Use the Karman–Howarth equation,

$$\frac{\partial}{\partial t} \langle \mathbf{u} \cdot \mathbf{u}' \rangle = \frac{1}{r^2} \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} \left(r^4 u^3 K \right) + 2\,\nu \nabla^2 \langle \mathbf{u} \cdot \mathbf{u}' \rangle\,,$$

to show that

$$L = \text{constant}\,, \qquad \frac{dI}{dt} = 8\pi \left[r^4 u^3 K \right]_\infty - 12\nu L\,.$$

What is the physical interpretation of the conservation of L?

(iii) Consider the case where all two-point correlations decay exponentially fast for large r. Show that, in such a situation, $L = 0$ while I is an invariant, and hence derive Kolmogorov's decay laws. State any assumption which you make.

Use the identity

$$(\mathbf{x} \times \mathbf{u}) \cdot (\mathbf{x}' \times \mathbf{u}') + (\mathbf{x}' - \mathbf{x})^2 (\mathbf{u} \cdot \mathbf{u}') = \nabla \cdot \left[\left((\mathbf{x}')^2 - (\mathbf{x} \cdot \mathbf{x}') \right) (\mathbf{x} \cdot \mathbf{u}')\, \mathbf{u} \right] + \nabla' \cdot \left[\mathbf{x}^2\, (\mathbf{x}' \cdot \mathbf{u})\, \mathbf{u}' \right]$$

to show that, for turbulence confined to a large, closed domain,

$$\left\langle \left[\int \mathbf{x} \times \mathbf{u} dV \right]^2 \right\rangle = -\int\!\!\int r^2 \langle \mathbf{u} \cdot \mathbf{u}' \rangle \, d\mathbf{x}\, d\mathbf{x}'\,.$$

How did Landau use this expression to explain the conservation of I in the absence of long-range correlations? What are the weaknesses in Landau's argument?

(iv) Discuss Batchelor and Proudman's objections to the conservation of I. Explain briefly why they expected pressure-velocity correlations of the form $\langle u_x^2 p' \rangle_\infty$ to fall off as r^{-3} with separation r? What is the significance of this for the triple correlations, and hence for I? Why did these objections appeal to researchers engaged in developing heuristic two-point closure models? What do recent numerical simulations show regarding I?

2 (i) There are three canonical problems in passive scalar mixing: Taylor's problem, Richardson's problem, and the Kolmogorov-Obukhov-Corrsin problem. Discuss briefly what each associated theory sets out to achieve, distinguishing clearly among the three problems.

(ii) Consider statistically stationary, homogeneous turbulence. Let $\mathbf{X}(t)$ be the position of a particle released from $\mathbf{x} = 0$ at $t = 0$, and $\boldsymbol{v}(t)$ the Lagrangian velocity, $\boldsymbol{v}(t) = \mathbf{u}(\mathbf{X}(t), t)$. Show that

$$\frac{d}{dt}\langle \mathbf{X}^2 \rangle = 2\int_0^t \langle \boldsymbol{v}(t) \cdot \boldsymbol{v}(t-\tau) \rangle \, d\tau \, .$$

Let t_L be the Lagrangian autocorrelation time, defined by

$$\langle \mathbf{u}^2 \rangle t_L = \int_0^\infty \langle \boldsymbol{v}(t) \cdot \boldsymbol{v}(t-\tau) \rangle \, d\tau \, .$$

Show that

$$\langle \mathbf{X}^2 \rangle = \langle \mathbf{u}^2 \rangle t^2, \qquad t \ll t_L \, ,$$
$$\langle \mathbf{X}^2 \rangle = 2\langle \mathbf{u}^2 \rangle t_L t \, , \qquad t \gg t_L \, .$$

Give a physical interpretation of the $\langle \mathbf{X}^2 \rangle \sim t^2$ and $\langle \mathbf{X}^2 \rangle \sim t$ behaviour.

(iii) Consider a small cloud of pollutant of characteristic radius R diffusing in a field of homogeneous turbulence. Let ℓ and η be the integral and Kolmogorov scales of the turbulence, and suppose that $\eta \ll R \ll \ell$. Provide an argument to support the estimate

$$\frac{dR^2}{dt} \sim \epsilon^{1/3} R^{4/3} \, ,$$

where ϵ is the energy dissipation rate. What restrictions apply to this expression? Let $\delta\mathbf{x}$ be the instantaneous separation of two marked particles which are simultaneously released at $t = 0$ with initial separation $(\delta\mathbf{x})_0$. Show that, if the turbulence is statistically stationary, and $\eta^2 \ll \langle (\delta\mathbf{x})^2 \rangle \ll \ell^2$, then

$$\langle (\delta\mathbf{x})^2 \rangle = g \, \epsilon \, t^3 \, ,$$

for some constant g. Why might you expect g to be a universal constant? What do recent experiments and simulations suggest about g?

(iv) Let $C(\mathbf{x}, t)$ be the concentration field of a passive scalar, and let the datum for C be chosen such that $\langle C \rangle = 0$. The passive scalar evolves in a turbulent velocity field, $\mathbf{u}(x, t)$, and both $\mathbf{u}$ and C are statistically isotropic. Consider the structure function

$$\langle (\triangle C)^2 \rangle (r) \;=\; \langle (C' - C)^2 \rangle \,,$$

where $C' = C(\mathbf{x}')$, $C = C(\mathbf{x})$ and $\mathbf{r} = \mathbf{x}' - \mathbf{x}$. Show that

$$\langle (\triangle C)^2 \rangle \;=\; 2\langle C^2 \rangle \,, \qquad \text{for} \qquad r \to \infty \,,$$

and

$$\langle (\triangle C)^2 \rangle \;=\; \frac{1}{3}\,\epsilon_c r^2/\alpha \,, \qquad \text{for} \qquad r \to 0 \,,$$

where α is the diffusivity of the scalar and ϵ_c the scalar dissipation rate. We wish to determine $\langle (\triangle C)^2 \rangle$ for intermediate r.

Show that

$$\langle (\triangle C)^2 \rangle \sim \epsilon_c \epsilon^{-1/3}\, r^{2/3} \,, \qquad \hat{\eta} \ll r \ll \ell \,,$$

where $\hat{\eta} = \max\,(\eta, \eta_c)$ and η_c is the scalar microscale. State any assumptions that you make.

Consider the case of a weakly diffusive scalar, where $\alpha \ll \nu$ and $\eta_c \ll \eta$. In the viscous-convective range, $\eta_c \ll r \ll \eta$, the scalar is teased out by Kolmogorov sized eddies. Argue that $\frac{d}{dr}\langle (\triangle C)^2 \rangle$ depends only on ϵ_c, r and the strain-rate of these eddies and hence show that, in this range

$$\langle (\triangle C)^2 \rangle \sim \epsilon_c \sqrt{\frac{\nu}{\epsilon}}\; \ell n \left(\frac{r}{\eta_c} \right) \,.$$

Why must we use $\frac{d}{dr}\langle (\triangle C)^2 \rangle$, rather than $\langle (\triangle C)^2 \rangle$, in the development of this argument?

In the case of a highly diffusive scalar, $\alpha \gg \nu$, we have $\eta_c \gg \eta$. What is the form of $\langle (\triangle C)^2 \rangle$ in the inertial-diffusive range, $\eta \ll r \ll \eta_c$?

3 (i) Discuss how the second-order structure function, $\langle(\triangle v)^2\rangle(r)$, acts like a filter, distinguishing between the energy held above and below scale r. Explain the origin of the common, if simplistic, estimate

$$\frac{3}{4}\langle(\triangle v)^2\rangle(r) \sim \int_{\pi/r}^{\infty} E(k)\,dk\,.$$

(ii) Consider Kolmogorov's 1941 theory of the small scales. State Kolmogorov's first similarity hypothesis and deduce the associated forms of $\langle(\triangle v)^2\rangle$ and $E(k)$ in the universal equilibrium range. State Kolmogorov's second similarity hypothesis. What form does this impose on $\langle(\triangle v)^2\rangle$ and $E(k)$ in the inertial range?

(iii) Show that the exact relationship between $\langle(\triangle v)^2\rangle$ and $E(k)$ is

$$\frac{3}{4}\langle(\triangle v)^2\rangle(r) = \int_0^{\infty} E(k)\,H(kr)\,dk\,,$$

where

$$H(x) = 1 + 3\,x^{-2}\cos x - 3\,x^{-3}\sin x\,.$$

Given that a good approximation to $H(x)$ is

$$H(x) \approx (x/\pi)^2, \qquad x < \pi$$
$$H(x) \approx 1\ , \qquad x \geqslant \pi\,,$$

confirm that

$$\frac{3}{4}\langle(\triangle v)^2\rangle \approx \int_{\pi/r}^{\infty} E(k)\,dk + \frac{r^2}{\pi^2}\int_0^{\pi/r} k^2 E(k)\,dk\,.$$

Explain the physical origin of the second term on the right. It appears that $\langle(\triangle v)^2\rangle$ mixes information from different scales, and information about energy and enstrophy. Why does this pose a problem for Kolmogorov's two-thirds law?

(iv) Explain the basis of Kolmogorov's refined similarity hypothesis and show that it demands

$$\langle(\triangle v)^p\rangle = \beta_p\langle\epsilon_{AV}^{p/3}(r)\rangle\, r^{p/3}\,,$$

where p is an integer, ϵ_{AV} is the dissipation averaged over scale r and β_p are universal constants. Explain briefly how Kolmogorov used this expression to estimate the scaling exponent ζ_p in the expression $\langle(\triangle v)^p\rangle \sim r^{\zeta_p}$.

(v) Construct an argument which suggests that, like $\langle(\triangle v)^2\rangle$, higher-order structure functions mix information from different scales. What is the implication of this for the 1962 theory?

END OF PAPER