UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2008 9.00 to 12.00

PAPER 83

SOFT MATTER

There are ***FOUR*** *questions in total.*

Attempt questions one ***AND*** *two,*
and ***EITHER*** *question three* ***OR*** *four.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A twisted nematic display device is constructed from two plates parallel to the xy plane, one intersecting $z = 0$, the second at $z = L$, with a nematic liquid crystal in between. Let the director $\mathbf{n}$ be described by polar angles $\theta(x, y, z)$ and $\phi(x, y, z)$, so $\mathbf{n} = [\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta]$. Assume the surfaces of the plates touching the nematic are treated so the boundary conditions on the director are $\phi(x, y, 0) = 0$, $\phi(x, y, L) = \pi/2$, forcing it to undergo a rotation of $\pi/2$ from the bottom plate to the top plate. In the presence of an electric field the director will be a minimiser of the Franck elastic energy

$$F = \frac{1}{2}\int d^3x \left\{K_1(\nabla\cdot\mathbf{n})^2 + K_2[\mathbf{n}\cdot(\nabla\times\mathbf{n})]^2 + K_3[\mathbf{n}\times(\nabla\times\mathbf{n})]^2 - \frac{\epsilon}{8\pi}(\mathbf{E}\cdot\mathbf{n})^2\right\} ,$$

where K_i are the elastic constants. and ϵ is the dielectric constant anisotropy. Explain the physical meaning of the three elastic terms in this functional.

Suppose the electric field is $\mathbf{E} = (0, 0, E_0)$, normal to the plates. Show that the free energy density has the form

$$\frac{1}{2}f(\theta)\left(\frac{d\theta}{dz}\right)^2 + \frac{1}{2}g(\theta)\left(\frac{d\phi}{dz}\right)^2 - \frac{\epsilon E^2}{8\pi}\sin^2\theta , \qquad (1)$$

and find $f(\theta)$ and $g(\theta)$.

Find the variational (Euler-Lagrange) equations corresponding to (1), and verify that in the absence of the field the twisted profile $\theta = 0$, $\phi(z) = \pi z/2L$ is a solution.

Assuming that near the onset of a field-induced transition the angle θ is everywhere small, and vanishes on the top and bottom plates, one expects it to take the form $\theta(z) \simeq \theta_0\sin(\pi z/L)$. Use this trial solution in the Euler-Lagrange equations to show that the critical voltage for the onset of an instability is

$$V_c = 2\pi^{3/2}\left[K_1 + \frac{1}{4}(K_3 - 2K_2)\right]^{1/2}\epsilon^{-1/2} .$$

The resulting director profile is useful in display technology. Practical devices have plates which are crossed polarizers, and a mirror sits beneath the bottom plate. L is taken much greater than the wavelength of light, so the plane of polarisation adiabatically follows the director field. Explain how the application of a voltage allows this display technology to function.

2 Consider the Langevin equation for a single particle of mass m, drag coefficient γ and random forcing $\mathbf{A}'(t)$,

$$m\,\frac{d\mathbf{u}}{dt} \;=\; -\gamma\mathbf{u} + \mathbf{A}'(t)\;. \tag{1}$$

Assume the random force has zero mean and a variance $< \mathbf{A}'(t)\cdot\mathbf{A}'(t') >$ that is a function $\phi(|t-t'|)$ decaying very rapidly with $t-t'$, satisfying $\int_{-\infty}^{\infty} dy\phi(y) = m^2\tau$. If $\mathbf{u}(0) = \mathbf{u}_0$ and $\mathbf{r}(0) = \mathbf{r}_0$ are the initial velocity and position, solve (1) to obtain $\mathbf{U} \equiv \mathbf{u}(t) - \mathbf{u}_0 \mathrm{e}^{-\zeta t}$ formally in terms of $\mathbf{A}$, where $\zeta = \gamma/m$ and $\mathbf{A} = \mathbf{A}'/m$. From this deduce the variance $\langle U^2\rangle$ and thereby determine τ from equipartition.

In order to evaluate higher moments of $\mathbf{U}$, assume that the random process $A(t)$ is Gaussian, so

$$\begin{aligned}\langle A(t_1)A(t_2)\cdots A(t_{2n+1})\rangle &= 0\\ \langle A(t_1)A(t_2)\cdots A(t_{2n})\rangle &= \sum_{\text{all pairs}} \langle A(t_i)A(t_j)\rangle\langle A(t_k)A(t_l)\rangle\cdots\end{aligned}$$

Considering carefully the number of pairs in the above sum, show that the moments satisfy

$$\begin{aligned}\langle U^{2n+1}\rangle &= 0\\ \langle U^{2n}\rangle &= (2n-1)!!\langle U^2\rangle^n\end{aligned}$$

where $(2n-1)!! = (2n-1)(2n-3)\ldots 1$, and hence that the probability distribution of $\mathbf{U}$ is Gaussian,

$$W(\mathbf{u},t;\mathbf{u}_0) = \left[\frac{m}{2\pi k_B T(1-\mathrm{e}^{-2\zeta t})}\right]^{3/2} \exp\left[-\frac{m|\mathbf{u}-\mathbf{u}_0\mathrm{e}^{-\zeta t}|^2}{2k_B T(1-\mathrm{e}^{-2\zeta t})}\right]\;.$$

Integrate the equation for $\mathbf{u}$ to obtain the position vector $\mathbf{r}$. Find the mean and variance of $\mathbf{r}$. Examine the short and long-time behaviour and explain the distinction between the two.

3 Consider a semi-infinite elastic filament lying along the positive x-axis. The approximate equation of motion for small-amplitude perturbations $y = h(x,t)$ to the straight equilibrium is

$$\zeta h_t = -A h_{xxxx} ,$$

where ζ is the perpendicular drag coefficient and A is the bending modulus. Suppose the filament end at the origin is forced up and down so that the boundary conditions are $h(0,t) = h_0 \cos(\omega t)$ and $h_{xx}(0,t) = 0$, while, as $x \to \infty$, h and its derivatives vanish.

(a) Using dimensional analysis, find a characteristic length scale $\ell(\omega)$ for this problem in terms of A, ζ, and ω.

(b) Show that the solution to this boundary-value problem can be obtained by separation of variables,

$$h(x,t) = h_0 \Re\left\{ \mathrm{e}^{i\omega t} F(\eta) \right\} ,$$

where $\eta = x/\ell(\omega)$. Find the ode obeyed by F.

(c) Find the two solutions of the characteristic equation that satisfy the boundary conditions on F at $x = \infty$, and the amplitudes of those terms that satisfy the boundary conditions at $x = 0$. Show therefore that $h(x,t)$ is a superposition of two travelling waves which decay with increasing x, one moving to the left, and one to the right. Which one dominates?

4 A wormlike polymer of contour length L is subject to an external force f acting at its two ends, directed along the z axis. The effective energy is

$$\mathcal{E} = \frac{1}{2} A \int_0^L ds\, \kappa^2 - fz \ ,$$

where A is the bending modulus, z is the end-to-end extension, and ds is the differential of arclength s. Consider the high-force limit, where the chain's configuration deviates only slightly from a straight line. Then the tangent vector $\hat{\mathbf{t}}$ fluctuates only slightly around $\hat{\mathbf{z}}$, the unit vector in the z direction. If we take t_x and t_y as independent fluctuating components, the constraint $|\hat{\mathbf{t}}| = 1$ shows that t_z deviates from unity quadratically in the vector $\mathbf{t}_\perp \equiv (t_x, t_y)$. Show that to quadratic order the energy is

$$\mathcal{E} = \frac{1}{2} \int ds \left[A(\partial_s \mathbf{t}_\perp)^2 + f\mathbf{t}_\perp^2 \right] - fL \ .$$

Use equipartition to find the thermal average $\langle \mathbf{t}_\perp^2 \rangle$, being careful to account for the two independent components of $\mathbf{t}_\perp$. From this, show that in this high-force limit the force-extension relation takes the form

$$\frac{z}{L} = 1 - \sqrt{\frac{k_B T}{4 L_p f}} \ .$$

Compare this asymptotic result with that for the freely-jointed chain composed of N links, each of length b.

Calculate the correlation function $C(y) = \langle (1/L) \int_0^L ds \mathbf{t}_\perp(s) \cdot \mathbf{t}_\perp(s+r) \rangle$ of the tangent vector and thereby find the correlation length ξ, the length scale for decay of $C(y)$.

END OF PAPER