UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Tuesday 3 June 2008 1.30 to 4.30

PAPER 82

PERTURBATION AND STABILITY METHODS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Find three terms of an asymptotic expansion for each root of the equation

$$\epsilon x^3 + x^2 + 2x + 1 = 0$$

in the limit $\epsilon \to 0$.

(b) The (Legendre) function $P_n(x)$ is defined for $x \geqslant 1$ as

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left[x + \left(x^2 - 1\right)^{1/2} \cos\theta \right]^n d\theta \, .$$

In the limit $n \to \infty$ find leading order asymptotic approximations for (i) $P_n(x) \quad x > 1$; (ii) $P_n(1)$.

Deduce that for $x \to 1$ there is a distinguished scaling

$$x = 1 + \nu / n^q$$

where q should be determined, and find, in the form of an integral, the leading order asymptotic approximation for $P_n(x)$ when $n \to \infty$ with ν fixed.

Verify that this result agrees in an appropriate sense with both (i) and (ii).

Find asymptotic approximations for $P_n'(1)$ and $P_n''(1)$ as $n \to \infty$ and give a sketch of $P_n(x)$ for large n.

[*If you quote a standard result for the asymptotic approximation of an integral, a* **brief** *derivation of the result should be given.*]

2 Explain briefly the class of problems to which it is appropriate to apply the method of multiple scales.

Consider the Mathieu equation

$$\ddot{y} + \left(\omega^2 + \epsilon \cos t\right) y = 0\,, \quad t \geqslant 0\,,$$

in which $\omega > 0$ is a constant of order unity and $\epsilon \ll 1$ is a constant. The aim of the question is to determine the values of ω, if any, for which the equation has a growing solution for $t \to \infty$.

(i) Suppose that $\omega = \frac{1}{2} + k\epsilon$.

Use the method of multiple scales to find a general solution for $y(t)$ that is valid for times $t = \mathrm{ord}(\epsilon^{-1})$. Deduce the range of values of k for which the equation has a growing solution.

(ii) Now consider other values for ω and the task of determining solutions for $y(t)$ that are valid when $t = \mathrm{ord}(\epsilon^{-2})$. Explain why the equation may have a growing solution if $\omega = 1$. Explain briefly why there is an interval of size $\mathrm{ord}(\epsilon^p)$, where p should be specified, containing this value of ω for which solutions of the equation can grow. Use multiple scales to find this interval explicitly.

(iii) Generalise these results to suggest, *without detailed calculation*, all the values of $\omega > 0$ for which the equation has a growing solution as $t \to \infty$.

[*You are* **not** *required to verify your answer in part (iii).*]

3 The function $f(r,\epsilon)$ satisfies the equation

$$f_{rr} + \frac{3}{2r} f_r + \epsilon f f_r = 0 \quad \text{in} \quad r \geqslant 1\,,$$

where $0 < \epsilon \ll 1$. The function $f(r,\epsilon)$ also satisfies the boundary conditions

$$f = 0 \quad \text{at} \quad r = 1, \quad \text{and} \quad f \to 1 \quad \text{as} \quad r \to \infty\ .$$

For both $r = \text{ord}(1)$ and $r = \text{ord}(\epsilon^{-1})$ obtain asymptotic expansions for f up to and including $O(\epsilon)$ terms.

Hint. You may quote the general solution $y(x)$ of

$$\left(x^{3/2} e^{x} y'\right)' = E_{3/2}(x) \equiv \int_x^\infty t^{-3/2} e^{-t}\, dt\ ,$$

with $y \to 0$ as $x \to \infty$, as $y(x) = \beta E_{3/2} + G(x)$, where β is an arbitrary constant and $G(x)$ is a function such that as $x \to 0$

$$y = \beta\left(2x^{-1/2} - 2\sqrt{\pi} + 2x^{1/2} + O(x^{3/2})\right) + \left(4\ln x + \mu + O(x^{1/2})\right)\ ,$$

where μ is a constant.

4 Rayleigh's equation governing the linear inviscid instability of an unidirectional flow $(U(y), 0, 0)$, subject to a 2D disturbance with wavenumber k and complex wavespeed $c = c_r + ic_i$, is

$$(U - c)(\phi'' - k^2\phi) - U''\phi = 0\,,$$

where $(\widetilde{u}, \widetilde{v}, 0) = (\phi', -ik\phi, 0)$ and $ik\widetilde{p} = -ik(U - c)\widetilde{u} - U'\widetilde{v}$ are the perturbation velocity and pressure respectively. Assume boundary conditions $\widetilde{v} = 0$ on $y = \pm y_0$.

(a) Show that

$$\frac{d}{dy}\left[(U - c)^2\,\frac{d\psi}{dy}\right] - k^2\,(U - c)^2\,\psi = 0\,,$$

where $\psi \equiv \phi/(U - c)$. Deduce that

$$\int_{-y_0}^{y_0} U\left(|\psi'|^2 + k^2|\psi|^2\right) dy = \int_{-y_0}^{y_0} c_r\left(|\psi'|^2 + k^2|\psi|^2\right) dy\,,$$

and hence that if $c_i \neq 0$ the real part of c, i.e. c_r, is bounded by the maximum and minimum values of $U(y)$, say U_{max} and U_{min} respectively. By considering

$$\int_{-y_0}^{y_0} (U - U_{max})(U - U_{min})\left(|\psi'|^2 + k^2|\psi|^2\right) dy\,,$$

or otherwise, deduce that

$$\left(c_r - \tfrac{1}{2}(U_{max} + U_{min})\right)^2 + c_i^2 \leqslant \left(\tfrac{1}{2}(U_{max} - U_{min})\right)^2\,,$$

and give a graphical interpretation of this result in the c-plane.

(b) Suppose that $y_0 = \infty$ and

$$U(y) = \begin{cases} 1 - |y| & \text{if } |y| \leqslant 1\,, \\ 0 & \text{if } |y| \geqslant 1\,, \end{cases}$$

and assume that $\widetilde{v}$ and $\widetilde{p}$ are everywhere continuous. Show that the eigenvalue relation for the mode with ϕ even (the so-called *sinuous mode*) is

$$2k^2c^2 + k(1 - 2k - e^{-2k})c - (1 - k - (1 + k)e^{-2k}) = 0\,,$$

and that for the mode with ϕ odd (the so-called *varicose mode*) is

$$2kc - (1 - e^{-2k}) = 0\,.$$

Briefly discuss whether there are unstable modes.

END OF PAPER