UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2008 9.00 to 12.00

PAPER 81

FUNDAMENTALS OF ATMOSPHERE–OCEAN DYNAMICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

Clarity and explicitness of reasoning will attract more credit than perfection of computational detail.

(x, y, z) denotes right-handed Cartesian coordinates and (u, v, w) the corresponding velocity components; t is time; the gravitational acceleration is $(0, 0, -g)$ where g is a positive constant. The fluid is always incompressible. 'Ideal fluid' always means that buoyancy diffusion can be neglected where relevant, as well as viscosity. N denotes the buoyancy frequency of a stratified fluid.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State what is meant by the Boussinesq approximation for a continuously stratified ideal fluid, and what is meant by the 'buoyancy acceleration'. Define the buoyancy frequency $N(z)$.

Write down the nonlinear Boussinesq momentum, buoyancy and mass-conservation equations for a non-rotating reference frame. Show from the equations that, provided the pressure field satisfies a certain condition, which you should specify,

$$\frac{\partial \bar{u}}{\partial t} \;=\; -\frac{\partial}{\partial y}(\overline{uv}) - \frac{\partial}{\partial z}(\overline{uw})$$

where the overbars denote an x-average whose definition you should also specify.

Two-dimensional internal gravity waves are generated at a lower boundary $z = 0$ by forcing the vertical velocity field $w(x, z, t)$ to satisfy

$$w(x, 0, t) \;=\; a \exp(ikx - i\omega t) \qquad \text{(real part understood)} \tag{$*$}$$

where a, k and ω are real constants, and where a is sufficiently small for linearized theory to apply. Show from the linearized equations for two-dimensional motion that $w(x, z, t)$ satisfies

$$\nabla^2 w_{tt} + N^2 w_{xx} \;=\; 0\,, \tag{$\dagger$}$$

where suffixes denote partial differentiation. Is this equation valid when N varies with z? When N is constant, with $0 < \omega < N$, find two solutions of ($\dagger$) that satisfy the boundary condition ($*$). Explain briefly what is meant by the *radiation condition*, and show how it selects just one out of these two solutions.

Justify the radiation condition by making a an appropriate function of time in ($*$). Assume that a vanishes for all negative times but grows slowly toward a small constant value a_∞. Introduce a small parameter $\mu \ll 1$ and rescaled variables $T = \mu t$ and $Z = \mu z$, writing $a = a(T)$ in ($*$). Where a occurs in the solution for the wavefield, make it a function of (Z, T) and show that ($\dagger$) is satisfied correct to first order in μ provided that $a(Z, T)$ satisfies an equation of the form

$$\frac{\partial a}{\partial T} \;+\; b\,\frac{\partial a}{\partial Z} \;=\; 0$$

where b is a constant to be determined, and where the symbols $\partial a/\partial T$ and $\partial a/\partial Z$ have a meaning that you should explain carefully. Explain how the radiation condition is related to the sign of b.

Use the above to find an expression involving $a(Z, T)$ that describes, correct to $O(a^2)$, how the mean flow $\bar{u}$ evolves as the waves propagate away from $z = 0$. Explain briefly why the mean-flow changes would become an essential part of the energy budget if the system were to be viewed in a moving frame of reference, translating in the x direction with velocity ω/k.

2 By writing down the relevant equations in Cartesian components, demonstrate the partial analogy between stratified and rotating ideal-fluid flow. In particular, show that in Cartesian components the equations for stratified, non-rotating motion of an ideal fluid are mathematically identical to the equations for a non-stratified, rotating ideal fluid, under certain assumptions to be specified. Carefully indicate which quantity in the stratified system corresponds to which in the rotating system. Explain briefly why a free-surface, zero-pressure boundary condition would destroy the analogy. Show that the analogy remains valid if prescribed body forces are introduced into both problems, provided that the forces act in certain directions to be determined.

Now restrict attention to the stratified problem. Consider a steady, horizontally-directed weak body force of the form $\mathbf{F} = \big(F(x,z),\,0,\,0\big)$ with

$$F(x,z) \;=\; f'(x)\exp(imz) \qquad \text{(real part understood)} \qquad (*)$$

where m is a positive constant and where $f'(x)$ is real valued, taking positive values in a finite interval $|x| < x_1$ and zero values elsewhere; $f(x)$ is defined to be zero for $x < -x_1$ and to take a constant positive value ϵ for $x > x_1$. Assume that ϵ is small enough that the resulting fluid motion satisfies the equations linearized about rest. Show that the linearized equations have steady solutions with x-independent velocity field $\mathbf{u} = \big(u(z),0,0\big)$ and pressure-anomaly field

$$p \;\propto\; f(x)\exp(imz) + b(z) \qquad (\dagger)$$

where $b(z)$ is an arbitrary function.

Briefly consider a time-dependent version of the same problem, in which the force $\mathbf{F}$ is brought to its steady value $(*)$ from zero at some finite time in the past. Before that time, all the disturbance fields are zero everywhere. Do not solve this problem in detail, but use your knowledge of its general properties. Show that for fixed x the pressure-anomaly field p must, at sufficiently large time t, be given by $(\dagger)$ with a particular choice of $b(z)$ to be determined. How large is "sufficiently large"?

Now consider another stratified problem in which $\mathbf{F} = \big(F(x,y),\,0,\,0\big)$ where, with the same $f(x)$ as before,

$$F(x,y) \;=\; f'(x)\exp(i\ell y) \qquad \text{(real part understood)}$$

and where ℓ is another positive constant. Show that the linearized equations can no longer be satisfied by any steady, x-independent velocity field, or indeed by any steady velocity field whatever. Show that they do, however, allow unsteady velocity fields of the form

$$\mathbf{u} \;=\; (-t\phi_y,\, t\phi_x,\, 0)$$

where suffixes denote partial differentiation and where $\phi = \phi(x,y)$, satisfying

$$\phi_{xx} + \phi_{yy} \;=\; -i\,\ell f'(x)\exp(i\ell y)\,.$$

In the limiting case $x_1 \to 0$, show that

$$\phi \;=\; \tfrac{1}{2} i\epsilon\exp(i\ell y)\exp(-\ell|x|)\ .$$

Would you expect the corresponding unstratified, rotating problem to have similar solutions, evanescing with $|x|$? Give reasons.

[*You may quote without proof any standard theorems on rotating flow in the limit of small Rossby number.*]

3 Write down the shallow-water momentum and mass-conservation equations in a frame of reference rotating with constant angular velocity $(0, 0, \frac{1}{2}f)$. Allow for a sloping bottom boundary $z = b(x, y)$, taking care to distinguish between the layer depth $h(x, y, t)$ and the free-surface elevation $\zeta(x, y, t)$. Derive the vertical component of the vorticity equation from the momentum equations, and show that Rossby's potential vorticity q_a/h is an exact material invariant, i.e. that

$$\frac{\mathcal{D}_H}{\mathcal{D}t}\left(\frac{q_a}{h}\right) \;=\; 0\;, \tag{$*$}$$

where $q_a = f + v_x - u_y$ and where $\mathcal{D}_H/\mathcal{D}t = \partial/\partial t + \mathbf{u}_H \cdot \nabla_H$, with $\mathbf{u}_H = (u, v, 0)$ and $\nabla_H = (\partial/\partial x,\, \partial/\partial y,\, 0)\,.$

Using the standard order-of-magnitude arguments, carefully derive the quasi-geostrophic counterpart to $(*)$, expressing everything in terms of an appropriate streamfunction $\psi(x, y, t)$ and carefully explaining the assumptions made. What types of wavemotion described by the full equations are (a) excluded from, and (b) included in, the dynamics described by the quasi-geostrophic equations?

The fluid layer is confined to a channel $0 < y < L$ with a sloping bottom $b = \epsilon y$, where the channel width L is not much greater than the Rossby deformation length L_D and where ϵ is a positive constant, small enough for consistency with the assumptions of quasi-geostrophic theory. Linearizing about a state of relative rest in which the layer depth is h_{00} in mid-channel, find the x-periodic wave solutions that satisfy the boundary conditions $\psi = 0$ on $y = 0$ and $y = L$.

Sketch the shape of a contour of constant quasi-geostrophic potential vorticity, and use it to explain carefully how the wave propagation mechanism can be understood in terms of potential-vorticity advection and inversion.

The layer is now further confined by boundaries $x = 0,\, L$, so that it is confined to a square basin. By using the transformation

$$\hat{\psi} \;=\; \psi\, \exp(\tfrac{1}{2} i\beta x/\omega) \qquad (\beta = \epsilon f/h_{00})$$

or otherwise, find the frequencies ω of normal modes of the basin with time dependence $\exp(-i\omega t)$. Show that in the limit $L/L_D \to 0$

$$\omega \;\to\; \frac{\beta L}{2\pi(m^2 + n^2)^{1/2}}$$

where m and n are positive integers.

4 Consider quasi-geostrophic motion of a stratified, rotating, Boussinesq fluid on a β-plane. You may assume that the motion is governed by the quasi-geostrophic potential-vorticity equation

$$\frac{\mathcal{D}_{\rm g}Q}{\mathcal{D}t} \;=\; 0 \quad \text{with} \quad Q \;=\; \left[f_0 \;+\; \beta y \;+\; \frac{\partial^2\psi}{\partial x^2} \;+\; \frac{\partial^2\psi}{\partial y^2} \;+\; \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\,\frac{\partial\psi}{\partial z}\right)\right]$$

in the standard notation of the lecture notes, with velocity field

$$(u,\,v,\,w) \;=\; \left\{-\frac{\partial\psi}{\partial y}\;,\;\frac{\partial\psi}{\partial x}\;,\;-\frac{f_0}{N^2}\frac{\mathcal{D}_{\rm g}}{\mathcal{D}t}\left(\frac{\partial\psi}{\partial z}\right)\right\}\;.$$

Write down the definition of the symbol $\mathcal{D}_{\rm g}/\mathcal{D}t$ in terms of ψ, and explain how it differs from the exact material derivative. Write down the leading-order balances in the momentum equations — do not justify these — and deduce that the buoyancy-acceleration anomaly σ is equal to $f_0\partial\psi/\partial z$.

Let overbars denote averages in the x direction and primes disturbances about such averages so that, for instance, $\psi = \bar{\psi}(y,z) + \psi'(x,y,z,t)$. Without assuming that the disturbances have small amplitudes, derive the Taylor identity

$$\nabla\cdot(F,G) \;=\; \overline{v'Q'} \qquad (*)$$

where ∇ is in the yz plane and where (F,G) is defined by

$$(F,G) \;=\; \left(-\overline{u'v'},\;\frac{f_0}{N^2}\,\overline{v'\sigma'}\right)\;. \qquad (\dagger)$$

If furthermore the disturbance amplitude is small, deduce from the linearized potential-vorticity equation that the quantity $\mathcal{A} = \frac{1}{2}\overline{Q'^2}/\overline{Q}_y$ then satisfies

$$\frac{\partial\mathcal{A}}{\partial t} \;+\; \nabla\cdot(F,G) \;=\; 0\;. \qquad (\ddagger)$$

Consider a flow with constant N confined between a rigid, flat upper boundary $z = H$ and a rigid, slightly-undulating lower boundary $z \;=\; \mathrm{Re}\{\,\epsilon\,H\,e^{ikx}\,\sin(\ell y)\}$, where ϵ, k, ℓ, and H are real constants and where ϵ is assumed small enough to permit linearization of the equations. Take $\bar{\psi} = -\bar{u}y$ where

$$\bar{u} \;=\; \beta/\kappa^2\;, \qquad \kappa^2 \;=\; k^2+\ell^2+(f_0 m/N)^2$$

with $m = \pi/H$. Show that the linearized equations and boundary conditions admit a resonantly-growing solution

$$\psi' = \mathrm{Re}\left[\,\epsilon\,e^{ikx}\,\sin(\ell y)\big\{B\,t\,\cos(mz) \;+\; C\,(z-H)\sin(mz)\big\}\right]$$

where $B = 2ikf_0\bar{u}^2/\beta$ and $C = N^2/(f_0 m)$. For this solution evaluate all the quantities appearing in $(*)$, $(\dagger)$ and $(\ddagger)$, thereby verifying that $(*)$ and $(\ddagger)$ are indeed satisfied.

[*Show first that* $\overline{u'v'} = 0$, *that* $\partial\mathcal{A}/\partial t \propto B^2$, *and that the only terms contributing to* $(*)$ *are those* $\propto BC$.]

Now suppose that the wavemotion takes place in a channel $0 < y < \pi/\ell$ and that the waves grow to large amplitude and break, causing the mean potential vorticity $\bar{Q}$ to be mixed to a uniform value across the channel at all levels. Assume that the resulting change in $\bar{u}$ is a function of y only that vanishes at $y = 0,\,\pi/\ell$. Find that function and deduce that the total momentum change is $-\frac{1}{12}\beta(\pi/\ell)^2$ per unit mass.

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