MATHEMATICAL TRIPOS **Part III**

Wednesday 4 June 2008 1.30 to 4.30

PAPER 80

NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Attempt no more than **FOUR** *questions.*

There are **SIX** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a two-step method for ODEs defined by the polynomials

$$\rho(w) = w^2 - (1+\alpha)w + \alpha \qquad \text{and} \qquad \sigma(w) = \frac{1}{12}(5+\alpha)w^2 + \frac{2}{3}(1-\alpha)w - \frac{1}{12}(1+5\alpha),$$

where α is a real constant.

(a) Prove that for every α the method is of order $p \geqslant 3$ and that there exists unique value of α for which the method is of order 4.

(b) We know from the second Dahlquist barrier that the method cannot be A-stable for any value of α. Prove a stronger statement: for every α for which the method is convergent, its linear stability domain is necessarily a bounded subset of $\mathbb{C}$.

2 We are given the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\kappa \frac{\partial u}{\partial x},$$

where $\kappa \in \mathbb{R}$, together with initial conditions for $t = 0$, $0 \leqslant x \leqslant 1$, and zero boundary conditions at $x = 0, 1$, $t > 0$.

(a) Prove that the equation is well posed for all values of κ.

(b) The equation is semidiscretized by the method

$$u_m' = \frac{1}{(\Delta x)^2}(u_{m-1} - 2u_m + u_{m+1}) + \frac{\kappa}{\Delta x}(u_{m+1} - u_{m-1}), \qquad m = 1, \ldots, M,$$

where $\Delta x = 1/(M+1)$. Using the energy method, or otherwise, prove that the method is stable.

3 (a) Define algebraic stability of Runge–Kutta methods.

(b) Let $b_1, \ldots, b_s \geqslant 0$ and suppose that the matrix M is positive semidefinite. Prove that the underlying Runge–Kutta method is algebraically stable.

(c) Show that the two-stage Gauss–Legendre method

$$\begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

is algebraically stable.

4 The advection equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \qquad x \in \mathbb{R}, \quad t \geqslant 0,$$

given as a Cauchy initial-value problem, is solved by the finite difference method

$$\tfrac{1}{2}\mu(1+\mu)u_{m-1}^{n+1} + (1+\mu)(2-\mu)u_m^{n+1} + \tfrac{1}{2}(1-\mu)(2-\mu)u_{m+1}^{n+1} = (2-\mu)u_m^n + (1+\mu)u_{m+1}^n$$

for all $m \in \mathbb{Z}$, $n \geqslant 0$, where μ is the (suitably defined) Courant number.

(a) Find the order of the method.

(b) Determine the range of μ for which the method is stable.

5 (a) Let $\mathcal{L}$ be a linear differential operator acting on space variables. Stating precisely all necessary definitions, prove that, subject to positive definiteness of $\mathcal{L}$, the differential equation $\mathcal{L}u = f$ is the Euler–Lagrange equation of the variational functional $I(v) = \langle \mathcal{L}v, v\rangle - 2\langle f, v\rangle$ and that the weak solution of this differential equation exists and is the unique minimum of I.

(b) Let $\mathcal{L} = -\nabla^2$ in the square $[0,1]^2$, given with zero boundary conditions. Prove that all the conditions required in part **(a)** are satisfied and derive the Ritz equations in a form suitable for the finite element method.

6 (a) Let A be a symmetric $m \times m$ matrix and denote by $\lambda_1, \ldots, \lambda_m$ its eigenvalues. We define the *spectral abscissa* as $\mu(A) = \max_{k=1,\ldots,m} \lambda_k$. Prove that (in Euclidean norm)

$$\|e^{tA}\| \leqslant e^{t\mu(A)}, \qquad t \geqslant 0$$

and that $\mu(A)$ is the smallest real number for which the above inequality is always correct.

(b) Let $\Phi(t) = e^{tA}e^{tB}$ be the Beam–Warming splitting of the matrix exponential $e^{t(A+B)}$. By considering the function $\Phi'(t) - (A+B)\Phi(t)$, or otherwise, prove that

$$\Phi(t) = e^{t(A+B)} + \int_0^t e^{(t-x)(A+B)}[e^{xA}, B]e^{xB}dx,$$

where $[\,\cdot\,,\,\cdot\,]$ is the matrix commutator.

(c) Suppose that both A and B are symmetric, negative-definite matrices. Prove that, in Euclidean norm,

$$\|\Phi(t) - e^{t(A+B)}\| \leqslant 2\|B\| \frac{e^{t[\mu(A)+\mu(B)]} - e^{t\mu(A+B)}}{\mu(A)+\mu(B)-\mu(A+B)},$$

provided that $\mu(A+B) < \mu(A) + \mu(B)$. What is the appropriate inequality when $\mu(A+B) = \mu(A) + \mu(B)$?

END OF PAPER