UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2008 1.30 to 4.30

# PAPER 78

# SYMMETRIC DYNAMICAL SYSTEMS

*Attempt no more than* ***THREE*** *questions*

*There are* ***FOUR*** *questions in total*

*The questions carry equal weight*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** A nonlinear oscillator is described by the second order ODE

$$\ddot{x} + q\dot{x} - x^2 + x\dot{x} + p = 0, \qquad (\dagger)$$

where $p$ and $q$ are parameters.

**(a)** Find the equilibria and the local bifurcation curves in the $(p, q)$ plane. Prove that no periodic orbits can exist when $p < 0$.

**(b)** Adopt the rescaling $x = \varepsilon^2 u$, $\frac{d}{dt} = \varepsilon \frac{d}{d\tilde{t}}$, $p = \varepsilon^4 \kappa$, $q = \varepsilon^2 \lambda$ and show that, for $\varepsilon = 0$, the rescaled system has a conserved quantity $H(u, \dot{u}) = \frac{1}{2}\dot{u}^2 + \kappa u - \frac{1}{3}u^3$ where $\dot{u} \equiv du/d\tilde{t}$. Sketch curves of constant $H$ in the $(u, \dot{u})$ plane and write down the value of $H$ corresponding to the homoclinic orbit.

**(c)** Explain briefly, without carrying out detailed calculations, how to deduce that a curve of global bifurcations exists close to $q = \frac{5}{7}\sqrt{p}$.

**(d)** Now consider the system

$$\begin{aligned} \ddot{x} + q\dot{x} - x^2 + x\dot{x} + z &= 0, \\ \dot{z} + z + z^3 - \dot{x} - p &= 0, \end{aligned}$$

which might describe a feedback loop providing indirect control of the parameter $p$. Show that $x = z = 0$ is a non-hyperbolic equilibrium point when $p = 0$.

**(e)** Assuming that $q \neq -1$, carry out an extended centre manifold reduction near $x = z = p = 0$ to determine the type of steady-state bifurcation which occurs at this point.

*Hint: it is sufficient to write* $\dot{x} \equiv y = a_1 p + a_2 x^2 + \cdots$; $\quad z = b_1 p + b_2 x^2 + \cdots$.

**2** Using the notation introduced below, this question considers the dynamics created near the Shilnikov bifurcation in $\mathbb{R}^3$. Consider a vector field whose linearisation around $(0,0,0)$ takes the form

$$\begin{aligned}\dot{x} &= \lambda_- x - \omega y,\\ \dot{y} &= \omega x + \lambda_- y,\\ \dot{z} &= \lambda_+ z,\end{aligned}$$

where $\omega > 0$ and $\lambda_- < 0 < \lambda_+$. Let the unstable manifold $W^u(0,0,0)$ intersect the plane $\Sigma$ given by $y = 0$ at the point $(r,\theta,z) = (\rho, 0, -\mu)$ in cylindrical polar coordinates.

**(a)** Derive an approximate two-dimensional return map $\Pi_S : \Sigma \to \Sigma$, stating your assumptions clearly.

**(b)** Justify the further approximation of your two-dimensional map by a one-dimensional map of the form

$$z_{n+1} = f_S(z_n) \equiv -\mu + A z_n^{\delta} \cos\left(\tfrac{\omega}{\lambda_+}\log(z_n) + \Phi\right), \qquad (*)$$

where $\delta = -\lambda_-/\lambda_+$, and $A$ and $\Phi$ are constants which may be assumed to be positive.

**(c)** Discuss the dynamics of the map $(*)$ in the two cases $\delta > 1$ and $\frac{1}{2} < \delta < 1$. In each case you should derive an expression relating $\mu$ and the period of any periodic orbits that exist.

**3** A particular representation of the group $\mathcal{G} = \mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$ on $\mathbb{R}^3$ is generated by

$$\rho = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \kappa_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\kappa_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \kappa_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

**(a)** Show that this representation of $\mathcal{G}$ is absolutely irreducible.

**(b)** Applying the Equivariant Branching Lemma, deduce the existence of two distinct axial branches of equilibria created in a generic $\mathcal{G}$-equivariant bifurcation.

**(c)** Show that, up to cubic order, the generic $\mathcal{G}$-equivariant amplitude equations may be written in the form

$$\begin{aligned} \dot{x} &= x(\mu - aX - cy^2 + ez^2), \\ \dot{y} &= y(\mu - aX - cz^2 + ex^2), \\ \dot{z} &= z(\mu - aX - cx^2 + ey^2), \end{aligned}$$

where $X = x^2+y^2+z^2$ and $\mu$, $a$, $c$ and $e$ are constants. You may assume for the remainder of the question that $\mu$ and $a$ are positive.

**(d)** Show, by direct calculation, that additional equilibria to those found in part (b) may exist when $ce < 0$. Explain why this does not contradict the Equivariant Branching Lemma. Without detailed calculation, describe the bifurcation that occurs at fixed $\mu > 0$ as either $c$ or $e$ changes sign.

**(e)** Let $V(x, y, z) = xyz$. By considering $dX/dt$ and $dV/dt$, or otherwise, show that there are no asymptotically stable equilibria when $c = e$. Sketch the behaviour of trajectories on the surface $X = \frac{\mu}{a}$ in this case.

**4** A $D_3$-symmetric collection of coupled oscillators undergoes a Hopf bifurcation with frequency $\omega_0$. The corresponding action of $D_3 \times S^1$ on the centre manifold $\mathbb{C}^2$ is generated by

$$\begin{aligned}
\rho(z_1, z_2) &= (\mathrm{e}^{-2\pi\mathrm{i}/3} z_1, \mathrm{e}^{2\pi\mathrm{i}/3} z_2), \\
m_x(z_1, z_2) &= (z_2, z_1), \\
\tau_\phi(z_1, z_2) &= \mathrm{e}^{\mathrm{i}\omega_0\phi}(z_1, z_2).
\end{aligned}$$

**(a)** State the Equivariant Hopf Theorem. Show that three distinct branches of periodic orbits bifurcate from the origin in a generic $D_3$ Hopf bifurcation with this group action.

**(b)** Derive the corresponding $D_3 \times S^1$-equivariant amplitude equations for $z_1$ and $z_2$, up to and including fifth-order terms. Hence show that periodic orbits on each branch have, generically, different amplitudes.

**(c)** Now consider the case where the rotation symmetry $\rho$ is weakly broken while the reflection symmetry $m_x$ and time translation $\tau_\phi$ are preserved. Show that additional linear terms now appear in the amplitude equations, and discuss the effect of these terms on the branches of periodic orbits identified in part (a). What effects does this weak symmetry breaking have on the bifurcation diagram?

**END OF PAPER**