UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 29 May 2008 1.30 to 3.30

PAPER 76

BRAIN IMAGING METHODS VIA ELECTRO- AND MAGNETO- ENCEPHALOGRAPHY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
Auxiliary Formulae data sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In the spherical model of the brain with a dipole excitation $(\mathbf{r}_0, \mathbf{Q})$, the exterior electric and magnetic potentials are given by

$$u^+(\mathbf{r}) = \frac{1}{4\pi\sigma}\mathbf{Q}\cdot\left[2\frac{\mathbf{P}}{P^3} + \frac{1}{rP}\frac{P\mathbf{r}+r\mathbf{P}}{Pr+\mathbf{r}\cdot\mathbf{P}}\right], \quad r > a$$

and

$$U(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{Q}\times\mathbf{r}_0\cdot\mathbf{r}}{rP^2+P\mathbf{r}\cdot\mathbf{P}}, \quad r > a$$

where $\mathbf{P} = \mathbf{r} - \mathbf{r}_0$ and a is the radius of the sphere. Find the singularities of u^+ and U in $\mathbb{R}^3$, as well as their order.

2 Use the Geselowitz formula for the magnetic field together with the potential representation

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla U(\mathbf{r}), \quad r > a$$

to construct a function ϕ in closed form such that

$$U(\mathbf{r}) = \mathbf{Q}\times\mathbf{r}_0\cdot\nabla_{\mathbf{r}_0}\phi(\mathbf{r}), \quad r > a$$

for a spherical conductor of radius a.

3 The interior electric potential for the spherical model of the brain with a dipole $(\mathbf{Q}, \mathbf{r}_0)$ is given by

$$u^-(\mathbf{r}) = \frac{1}{4\pi\sigma}\mathbf{Q}\cdot\left[\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} + \frac{\overline{r}}{a}\frac{\overline{\mathbf{r}}-\mathbf{r}_0}{|\overline{\mathbf{r}}-\mathbf{r}_0|^3}\right] + \frac{1}{4\pi\sigma}\mathbf{Q}\cdot\frac{1}{aR}\frac{R\overline{\mathbf{r}}+\overline{r}\mathbf{R}}{R\overline{r}+\overline{\mathbf{r}}\cdot\mathbf{R}}$$

where $\overline{\mathbf{r}} = (a/r)^2\mathbf{r}$ and $\mathbf{R} = \overline{\mathbf{r}} - \mathbf{r}_0$. Show that u^- can be interpreted in terms of images as

$$u^-(\mathbf{r}) = \Psi(\mathbf{r};\mathbf{r}_0) + \left(\frac{r}{a}\right)^3\Psi\left(\mathbf{r};\left(\frac{r}{a}\right)^2\mathbf{r}_0\right) + \frac{r}{a}\int_0^{\left(\frac{r}{a}\right)^2}\Psi(\mathbf{r};t\mathbf{r}_0)dt$$

where

$$\Psi(\mathbf{r};\mathbf{r}_0) = \frac{1}{4\pi\sigma}\mathbf{Q}\cdot\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}$$

4 Prove that the Kelvin transformation

$$\mathbf{r} \to \overline{\mathbf{r}} = \frac{a^2}{r^2}\mathbf{r}$$

leaves the operator $\mathbf{r} \times \nabla_{\mathbf{r}}$ invariant. Compute the effect of Kelvin's transformation on the following system of vector spherical harmonics:

$$\begin{aligned}
\mathbf{P} &= \hat{\mathbf{r}}Y \\
\mathbf{B} &= \frac{r}{\sqrt{n(n+1)}}\nabla Y \\
\mathbf{C} &= -\frac{r}{\sqrt{n(n+1)}}\hat{\mathbf{r}} \times \nabla Y
\end{aligned}$$

where Y is any scalar spherical harmonic and for convenience, the indices have been dropped.

5 Prove that the function $H_n(\mathbf{r})r^{-(2n+1)}$ is harmonic if and only if the nth degree homogeneous polynomial $H_n(\mathbf{r})$

is harmonic.

END OF PAPER

MATHEMATICAL TRIPOS, PART III

Paper 76: Brain Imaging Methods via Electro- and Magneto- Encephalography

Auxiliary Formulae data sheet

1.
$$F(\mathbf{r};\mathbf{r}_0) = r|\mathbf{r}-\mathbf{r}_0|^2 + |\mathbf{r}-\mathbf{r}_0|\mathbf{r}\cdot(\mathbf{r}-\mathbf{r}_0)$$

2.
$$\left(1-2\gamma p+p^2\right)^{-1/2} = \sum_{n=0}^{\infty} p^n P_n(\gamma)$$

P_n: Legendre polynomials, $|p|<1,\ |\gamma|\leqslant 1$

3.
$$\sum_{n=1}^{\infty}\frac{1}{n}p^n P_n(\gamma) = -ln\frac{\sqrt{1-2\gamma p+p^2}+1-\gamma p}{2}$$

4.
$$\sum_{n=0}^{\infty}\frac{1}{n+1}p^{n+1}P_n(\gamma) = ln\frac{\sqrt{1-2\gamma p+p^2}+p-\gamma}{1-\gamma}$$

5.
$$\triangle u(\mathbf{r}) = \left(\frac{\overline{r}}{a}\right)^5 \overline{\triangle}\left[\frac{a}{\overline{r}}u\left(\frac{a^2}{\overline{r}^2}\overline{\mathbf{r}}\right)\right]$$

6.
$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}$$

7.
$$\int\frac{dr}{|\mathbf{r}-\mathbf{r}_0|} = ln\left(|\mathbf{r}-\mathbf{r}_0| + \hat{\mathbf{r}}\cdot(\mathbf{r}-\mathbf{r}_0)\right)$$

8.
$$\frac{|\mathbf{r}-\mathbf{r}_0|\mathbf{r}+r\left(\mathbf{r}-\mathbf{r}_0\right)}{F\left(\mathbf{r};\,\mathbf{r}_0\right)} = \nabla_{\mathbf{r_0}}ln\frac{|\mathbf{r}-\mathbf{r}_0|}{F\left(\mathbf{r};\,\mathbf{r}_0\right)}$$

9.
$$P_0(\gamma)=1,\quad P_1(\gamma)=\gamma,\quad P_2(\gamma)=\frac{3\gamma^2-1}{2}$$

10.
$$u(\mathbf{r}) = \frac{1}{4\pi\sigma}\mathbf{Q}\cdot\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \frac{1}{4\pi}\oint_S u^-(\mathbf{r}')\hat{\mathbf{n}}'\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}ds(\mathbf{r}')\,,\quad \forall\mathbf{r}\notin S$$

11.

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \frac{\mu_0\sigma}{4\pi}\oint_S u^-(\mathbf{r}')\hat{\mathbf{n}}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} ds(\mathbf{r}')\,, \quad \forall \mathbf{r} \notin S$$

12.

$$\begin{aligned} \sigma \triangle u^-(\mathbf{r}) &= \nabla \cdot J^p(\mathbf{r})\,, \quad && \mathbf{r} \in \Omega \\ \partial_n u^-(\mathbf{r}) &= 0\,, && \mathbf{r} \in \partial\Omega \end{aligned}$$

13.

$$\begin{aligned} \nabla \cdot (f\mathbf{g}) &= (\nabla f) \cdot \mathbf{g} + f\nabla \cdot \mathbf{g} \\ \nabla \times (f\mathbf{g}) &= (\nabla f) \times \mathbf{g} + f\nabla \times \mathbf{g} \\ \nabla \times (\nabla \times \mathbf{g}) &= \nabla\nabla \cdot \mathbf{g} - \triangle\mathbf{g} \end{aligned}$$

14.

$$P_n\left(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}_0\right) = \frac{4\pi}{2n+1} \sum_{m=-n}^{m} \hat{Y}_n^m(\hat{\mathbf{r}}) \hat{Y}_n^{m^\star}(\hat{\mathbf{r}}_0)$$

$\hat{Y}_n^{m^\star}$: orthonormalised complex for of spherical harmonics.