UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 29 May 2008 9.00 to 12.00

PAPER 75

APPROXIMATION THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 a) For a set $\Phi = (u_0, \ldots, u_n)$ of real continuous functions on $[a, b]$, define when it is a Chebyshev system, and prove that Φ is a Chebyshev system if and only if for any distinct points $(x_i)_0^n$ from $[a, b]$ the determinant

$$D(x_0, x_1, \ldots, x_n) := \begin{vmatrix} u_0(x_0) & \cdots & u_n(x_0) \\ \cdots & \cdots & \cdots \\ u_0(x_n) & \cdots & u_n(x_n) \end{vmatrix}$$

does not vanish.

b) Prove that, for any distinct λ_i, the set $\Phi = (e^{\lambda_i x})_{i=0}^n$ is a Chebyshev system on any $[a, b]$.

c) Prove that, on the circle (i.e. on the period $\mathbb{T} = [-\pi, \pi)$), there is no Chebyshev system of *even* dimension.

2 State the Korovkin theorem on approximation of functions $f \in C\,[0, 1]$ by positive linear operators.

For $k \geq 3$, let $\mathcal{S}_k(\Delta_n)$ be a sequence of spline spaces of degree $k-1$ on the interval $[0, 1]$ with the knot-sequences

$$\Delta_n = \{\, t_1^{(n)} = \ldots = t_k^{(n)} = 0 < t_{k+1}^{(n)} \leq \ldots \leq t_n^{(n)} < t_{n+1}^{(n)} = \ldots = t_{n+k}^{(n)} = 1 \,\}$$

such that $|\Delta_n| := \max_i |t_{i+1}^{(n)} - t_i^{(n)}| \to 0 \quad (n \to \infty)$. Consider the Schoenberg-type operator

$$V_n : C[0, 1] \to \mathcal{S}_k(\Delta_n)\,, \qquad V_n(f, t) = \sum_{i=1}^n f(\tau_i^{(n)}) N_{i,n}(t)\,,$$

where $(N_{i,n})$ is the B-spline basis for $\mathcal{S}_k(\Delta_n)$ and $\tau_i^{(n)}$ are any points satisfying

$$t_i^{(n)} < \tau_i^{(n)} < t_{i+k}^{(n)}\,.$$

Using the Korovkin theorem prove that, for any $f \in C[0, 1]$, we have

$$\| V_n(f) - f \|_{C[0,1]} \to 0 \quad (n \to \infty)\,.$$

Hint. You may use the B-spline expansion of the monomials $t^m = \sum_{i=1}^n a_{m,i} N_i(t)$, $0 \leq t \leq 1$, where (suppressing the index n) the coefficients $a_{m,i}$ can be determined from the Marsden identity

$$(x - t)^{k-1} = \sum_{i=1}^n \omega_i(x)\, N_i(t)\,, \qquad t_k \leq t \leq t_{n+1}\,, \qquad \forall x \in \mathbb{R}\,.$$

3 State the Chebyshev alternation theorem on the element of best approximation to a function $f \in C[-1,1]$ from $\mathcal{P}_n$, the space of all algebraic polynomials of degree n.

Let

$$E_n(f) := \inf_{p_n \in \mathcal{P}_n} \|f - p_n\|_{C\,[-1,1]} .$$

It is clear that, for any $f \in C[-1,1]$, we have the inequality

$$E_{n-1}(f) \geq E_n(f) .$$

a) For every n, give an example of the function $f = f_n$ such that $E_{n-1}(f) = E_n(f)$.

b) Prove that, for $f(x) = e^x$, and for any n, we have

$$E_{n-1}(f) > E_n(f) ,$$

i.e., for such f the inequality is strict.

c) Show that, to each $f \in C\,[-1,1]$, there exists a system of points $(x_{n,k})$ such that if $\ell_n(f)$ is the interpolating polynomial to f at the nodes $x_{n,0}, x_{n,1}, \ldots, x_{n,n}$, then

$$\|\ell_n(f) - f\|_\infty \to 0 .$$

4 On $C(I)$, the space of continuous functions on $I = [-1,1]$, the "tilde" operator is given by the rule $\widetilde{f}(\theta) := f(\cos\theta)$.

a) Prove that

$$\omega(\widetilde{f}, t) \leq \omega(f, t) ,$$

where $\omega(f,t)$ is the first modulus of continuity.

b) State the first Jackson theorem for periodic functions and deduce (justifying each step) its analogue for approximation by algebraic polynomials of degree $\leq n$ on $[-1,1]$:

$$E_n(f) \leq c\,\omega\Big(f, \tfrac{1}{n}\Big) .$$

5 Given $\Delta = (t_i)_{i=1}^{n+k}$, let ω_i and ψ_i be polynomials in $\mathcal{P}_{k-1}$ defined as

$$\omega_i(x) := (x - t_{i+1}) \cdots (x - t_{i+k-1}), \qquad \psi_i(x) := \tfrac{1}{(k-1)!}\, \omega_i(x),$$

and let $(N_i)_{i=1}^{n}$ be the corresponding B-spline sequence. From the Marsden identity:

$$(x-t)^{k-1} = \sum_{i=1}^{n} \omega_i(x)\, N_i(t), \qquad t_k \le t \le t_{n+1}, \qquad \forall x \in \mathbb{R}$$

derive that any algebraic polynomial $p \in \mathcal{P}_{k-1}$ has the B-spline expansion

$$p\,(t) \;=\; \sum_{i=1}^{n} \lambda_i(p)\, N_i(t), \qquad t \in [t_k, t_{n+1}],$$

and express the functional $\lambda_i(p)$ in terms of p, ψ_i and $x \in \mathbb{R}$. Explain briefly why $\lambda_i(p)$ are independent of x.

Use the latter expansion to prove that, with $t_i^* = \frac{1}{k-1}(t_{i+1} + \cdots + t_{i+k-1})$, we have

$$p\,(t) \;=\; \sum_{i=1}^{n} p\,(t_i^*)\, N_i(t), \qquad \forall p \in \mathcal{P}_1 .$$

6 1) Let $\mathbb{X}$ be an inner product space with the scalar product $(\cdot,\cdot)$ and the norm $\|x\| := (x,x)^{1/2}$, and let $\mathcal{U}_n$ be an n-dimensional subspace.

a) Prove that $u^* \in \mathcal{U}_n$ is the best approximation to $x \in \mathbb{X}$ from $\mathcal{U}_n$ if and only if

$$(x - u^*, v) \;=\; 0 \qquad \forall v \in \mathcal{U}_n .$$

b) Let $(u_j)_{j=1}^{n}$ be a basis for $\mathcal{U}_n$ and let $G = ((u_i, u_j))_{i,j=1}^{n}$ be the corresponding Gram matrix. Prove that the elements of the Gramian inverse $G^{-1} = (b_{jk})$ are

$$b_{jk} \;=\; (\widehat{u}_j, \widehat{u}_k),$$

where $(\widehat{u}_k)$ is the dual basis, i.e., $(u_i, \widehat{u}_k) = \delta_{ik}$.

(*Hint.* Use the equality $\delta_{ik} \;=\; (G \cdot G^{-1})_{ik}$.)

2) Given $f \in C[0,1]$ and a basis (N_j) of the L_∞-normalized B-splines, let

$$P_{\mathcal{S}}(f) \;:= s^* = \sum_{j=1}^{n} a_j\, N_j$$

be the best spline approximation to f from $\mathcal{S} := \operatorname{span}(N_j)$ with respect to the L_2-norm; then $P_{\mathcal{S}}$ is also well defined as an operator from $C[0,1]$ onto $C[0,1]$.

Show that the max-norm of $P_{\mathcal{S}}$ satisfies the inequality

$$\|P_{\mathcal{S}}\|_\infty \;\le\; \|G^{-1}\|_{\ell_\infty}$$

where $G = (M_i, N_j)$ is the Gram matrix.

END OF PAPER