UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 4 June 2008 9.00 to 12.00

PAPER 74

STELLAR AND PLANETARY MAGNETIC FIELDS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A stratified Boussinesq fluid is permeated by an *oblique* uniform magnetic field $\mathbf{B}_0 = B_0\mathbf{c}$, $\mathbf{c} = (\sin\alpha, 0, \cos\alpha)$, where α $(0 < \alpha < \pi/2)$ is the inclination of $\mathbf{B}_0$ to the vertical. Show that the linearised dimensionless equations describing small perturbations to a horizontally stratified static state with a linear temperature gradient take the form

$$\begin{aligned}
\sigma^{-1}\dot{\mathbf{u}} &= -\nabla p' + R\theta\hat{\mathbf{z}} + Q\zeta\mathbf{c}\cdot\nabla\mathbf{b} + \nabla^2\mathbf{u},\\
\dot{\mathbf{b}} &= \mathbf{c}\cdot\nabla\mathbf{u} + \zeta\nabla^2\mathbf{b},\\
\dot{\theta} &= \mathbf{u}\cdot\hat{\mathbf{z}} + \nabla^2\theta,\\
\nabla\cdot\mathbf{u} &= \nabla\cdot\mathbf{b} = 0,
\end{aligned}$$

where the definitions of the dimensionless quantities appearing should be given.

Consider two-dimensional disturbances independent of y, and derive a relation for the growth rate s for disturbances having the wavelike form $\theta = \hat{\theta}e^{st+ikx+imz}$, etc. (Boundary conditions in z need not be applied). Give an expression for R as a function of k, m such that $s = 0$.

Show that for $m = 1$, the values k_c of k that minimise this critical value of R, and the corresponding values R_c of R take the forms

$$\begin{aligned}
k_c &= -\frac{1}{\sqrt{2}} + C_1Q, \quad R_c = \frac{27}{4} + 3Q(\cos\alpha - \frac{1}{\sqrt{2}}\sin\alpha)^2 + C_2Q^2, \quad Q \ll 1,\\
k_c &= -\mathrm{cot}\alpha + \frac{C_3}{Q}, \quad R_c = \frac{\mathrm{cosec}^6\alpha}{\mathrm{cot}^2\alpha} + \frac{C_4}{Q}, \quad Q \gg 1.
\end{aligned}$$

(It is not required to calculate the values of $C_1, \ldots, C_4$).

Show that there is a value of α such that R_c and k_c are independent of Q.

2 A 'cyclonic event' may be modelled by a velocity field of the form

$$\mathbf{u} = (0, rf(r), g(r)), \quad 0 < t < T$$

in cylindrical polar coordinates (r, ϕ, z), with $\mathbf{u} = 0$ for $t < 0$ and $t > T$. It may be assumed that $f, g \to 0$ as $r \to \infty$ in such a way that all relevant integrals converge.

Assume that at $t = 0$ there is a uniform magnetic field $B_0\hat{\mathbf{x}}$. We wish to calculate the magnetic field for $0 < t < T$, using the ansatz $\mathbf{B} = B\hat{\mathbf{z}} + \nabla \times (A\hat{\mathbf{z}})$, with $(A, B) = \mathrm{Im}(\hat{A}(r,t)e^{i\phi}, \hat{B}(r,t)e^{i\phi})$.

Ignoring diffusion, show that $\hat{A}, \hat{B}$ satisfy the equations

$$\frac{\partial \hat{A}}{\partial t} + if\hat{A} = 0, \qquad \frac{\partial \hat{B}}{\partial t} + if\hat{B} = i\frac{g'}{r}\hat{A}.$$

Solve these equations to find $A(r,T)$, $B(r,T)$. Hence find as an integral the x-component of the emf $\mathcal{E} = \int (\mathbf{u} \times \mathbf{B})_x \, d\mathbf{x}$, per unit length in z, where the integral is over the (x, y) plane. Calculate this quantity in the case $f(r) = g(r) = a^2 - r^2$, $r < a$, and $f, g = 0$, $r \geqslant a$, and sketch its dependence on T. Comment on the behaviour of the sign of $\mathcal{E}$. Explain qualitatively the effect of small diffusion on the solution and the emf.

3 A solenoidal velocity field consists of a simple shear flow $\Omega y\hat{\mathbf{x}}$ and a time dependent helical wave $\mathbf{u} = \mathrm{Re}\left(\mathbf{v}(t)e^{i\mathbf{k}(t)\cdot\mathbf{x}}\right)$, where $\mathbf{k} = (k_x, -\Omega t k_x, k_z)$, and k_x, k_z are constants.

(i) Verify that $\mathbf{u}$ satisfies the equation

$$\dot{\mathbf{u}} + \Omega y \frac{\partial \mathbf{u}}{\partial x} + \Omega u_y \hat{\mathbf{x}} = -\nabla p,$$

where $p = \mathrm{Re}(q(t)e^{i\mathbf{k}\cdot\mathbf{x}})$, with $q = 2i\Omega k_x v_y/|\mathbf{k}|^2$, provided that

$$\dot{\mathbf{v}} + \Omega v_y \hat{\mathbf{x}} = -i\mathbf{k}q,$$

and hence find an equation for the time-dependence of the quantity (related to the mean helicity) $H(t) \equiv ik_x(v_y v_z^* - v_z v_y^*)$.

(ii) The fluid is permeated by a uniform magnetic field $B_0\hat{\mathbf{x}}$. The flow induces a perturbation field $\mathbf{b}(\mathbf{x}, t)$. Using the First Order Smoothing approximation, show that the equation for $\mathbf{b}$ has the form

$$\dot{\mathbf{b}} + \Omega y \frac{\partial \mathbf{b}}{\partial x} = \Omega b_y \hat{\mathbf{x}} + B_0 \frac{\partial \mathbf{u}}{\partial x} + \eta \nabla^2 \mathbf{b}.$$

Show that this can also be solved in the form $\mathbf{b} = \mathrm{Re}\left(\mathbf{c}(t)e^{i\mathbf{k}(t)\cdot\mathbf{x}}\right)$,and derive the equation for the evolution of $\mathbf{c}$. Give an approximate expression for c_y, c_z when $\eta(k_x^2 + k_y^2) \gg |\Omega|$, and thus determine the x-component of the emf $\mathcal{E} \equiv \mathrm{Re}(v_y c_z^* - v_z c_y^*)$, in terms of H.

Calculate the time integral of $\mathcal{E}$, $\int_{-\infty}^{\infty} \mathcal{E} dt$, given that $H(0) = H_0$.

4 Write an essay on the solar cycle. Your essay should cover observational aspects, grand minima and the various theoretical scenarios for modelling the cycle.

END OF PAPER