MATHEMATICAL TRIPOS Part III

Friday 6 June 2008 9.00 to 11.00

PAPER 71

ACCRETION DISCS

There are **THREE** *questions in total.*
The questions carry equal weight.
Full marks can be obtained by completing **TWO** *questions.*

This is an **OPEN BOOK** *examination.*

Candidates may bring handwritten notes and lecture handouts into the examination.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider an accretion disc around a point mass, M. Matter is added to the disc at radius R_0 at a rate $\dot{M}(t)$. The specific angular momentum of the added material is $h_0 = (GMR_0)^{\frac{1}{2}}$. Derive the evolution equation for the surface density $\Sigma(R,t)$ for a given viscosity, ν.

Now consider the disc when it has reached a steady state with matter being added at a constant rate $\dot{M} = \text{const}$. At radius $R_{\text{in}} < R_0$ there is no viscous torque and matter leaves the disc at a rate $\dot{M}_{\text{in}}$. Similarly, at radius $R_{\text{out}} > R_0$, there is no viscous torque and matter leaves the disc at a rate $\dot{M}_{\text{out}}$. Show from your equations that $\dot{M} = \dot{M}_{\text{in}} + \dot{M}_{\text{out}}$.

Show also that

$$\frac{\dot{M}_{\text{in}}}{\dot{M}} = \frac{R_{\text{out}}^{\frac{1}{2}} - R_0^{\frac{1}{2}}}{R_{\text{out}}^{\frac{1}{2}} - R_{\text{in}}^{\frac{1}{2}}}.$$

2 **(a)** An accretion disc around a point mass is such that the predominant pressure is gas pressure and the main energy transfer is due to radiative diffusion. The opacity is of the form $\kappa \propto \rho^{-3/2}T^{5/4}$. Assuming the viscous parameter $\alpha = \text{const.}$, show that the viscosity ν obeys the relation

$$\nu \propto \Sigma^{1/2}R^{9/4}.$$

Comment on the thermal and viscous stability of the disc.

(b) Assume the viscosity takes the form

$$\nu = \nu_0 \left(\frac{\Sigma}{\Sigma_0}\right)^{1/2} \left(\frac{R}{R_0}\right)^{9/4},$$

where ν_0, Σ_0 and R_0 are constants. Show that the equation describing the evolution of such a disc can be written as

$$\frac{\partial S}{\partial \tau} = \frac{\partial^2}{\partial x^2}\left[S^{3/2}\,x\right],$$

where $\tau = 3\,\nu_0 t/4R_0^2$, $x^2 = R/R_0$ and

$$S(x,\tau) = \left(\frac{\Sigma}{\Sigma_0}\right)\left(\frac{R}{R_0}\right)^{3/2}.$$

Show that a solution of the equation is

$$S = (1-k\xi)^2/\tau,$$

where $\xi = x/\tau^{1/2}$ and k is a constant to be determined.

Give Σ explicitly as a function of R and τ, and describe and sketch the nature of the solution.

3 A spinning black hole with angular momentum $\mathbf{J}_h$ is at the centre of an accretion disc which has angular momentum $\mathbf{J}_d$. An elemental annulus of the disc, with angular momentum $\delta\mathbf{J}_d$, is subject to a Lense-Thirring torque $\delta\mathbf{T}$ such that

$$\delta\mathbf{T} \propto \mathbf{J}_h \wedge \delta\mathbf{J}_d.$$

Show that the total torque $\mathbf{T}$ acting on the disc must be perpendicular to $\mathbf{J}_h$.

A general form of the torque can be written as

$$\mathbf{T} \;=\; K_1 \left[\mathbf{J}_h \wedge \mathbf{J}_d\right] + K_2 \left[\mathbf{J}_h \wedge (\mathbf{J}_h \wedge \mathbf{J}_d)\right] \,,$$

where K_1 and K_2 are scalar quantities which depend on the properties of the disc. Write down the equation for $d\mathbf{J}_h/dt$, and explain briefly the physical effect of the K_1 and K_2 terms.

Show that $J_h = |\mathbf{J}_h|$ is constant.

Take $\mathbf{J}_t = \mathbf{J}_d + \mathbf{J}_h$ to be the total angular momentum of the disc plus black hole system. Show that

$$\frac{d}{dt}\left(\mathbf{J}_h \cdot \mathbf{J}_t\right) \;=\; K_2 \left[J_d^2 J_h^2 - \left(\mathbf{J}_d \cdot \mathbf{J}_h\right)^2\right] \;\equiv\; A\,.$$

Show also that

$$\frac{d}{dt}\left(\mathbf{J}_h \cdot \mathbf{J}_t\right) \;=\; \frac{d}{dt}\left(\mathbf{J}_h \cdot \mathbf{J}_d\right)\,,$$

and deduce that

$$\frac{d}{dt}\left(J_d^2\right) \;=\; -2A\,.$$

Assuming that $K_2 > 0$, deduce that in general $\mathbf{J}_h$ eventually aligns with $\mathbf{J}_t$.

Find the eventual values of $\mathbf{J}_h$ and $\mathbf{J}_d$ for the two initial configurations:

$$(a) \quad \mathbf{J}_h = \left(\frac{\sqrt{3}}{2}\,,\, 0\,,\, \frac{3}{2}\right) \;;\quad \mathbf{J}_d = \left(-\frac{\sqrt{3}}{2}\,,\, 0\,,\, \frac{1}{2}\right)$$

$$\text{and} \quad (b) \quad \mathbf{J}_h = \left(\frac{\sqrt{3}}{2}\,,\, 0\,,\, \frac{3}{2}\right) \;;\quad \mathbf{J}_d = \left(-\frac{\sqrt{3}}{2}\,,\, 0\,,\, -\frac{1}{2}\right).$$

Do the black hole and the disc always finish with their spins aligned?

END OF PAPER