UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2008 1.30 to 4.30

PAPER 70

STRUCTURE AND EVOLUTION OF STARS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You may use the equations and results given below without proof.

The symbols used in these questions have the meanings they were given in the lectures. Candidates are reminded of the equations of stellar structure in the form:

$$\frac{dm}{dr} = 4\pi r^2 \rho\,.$$

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,.$$

$$\frac{dL_r}{dr} = 4\pi r^2 \rho\epsilon\,.$$

In a radiative region

$$\frac{dT}{dr} = -\frac{3\,\kappa\rho L_r}{16\,\pi a c r^2 T^3}\,.$$

In a convective region

$$\frac{dT}{dr} = \frac{(\Gamma_2 - 1)T}{\Gamma_2 P}\,\frac{dP}{dr}\,.$$

The luminosity, radius, and effective temperature are related by

$$L = 4\pi R^2 \sigma T_e^4$$

The electron density for a fully ionized mixture of hydrogen and helium is

$$n = \frac{\rho}{(2m_H)}(1 + X)$$

The equation of state for an ideal gas and radiation is

$$P = \frac{\mathcal{R}\rho T}{\mu} + \frac{aT^4}{3}$$

with $1/\mu = 2X + 3Y/4 + Z/2$. *For an ideal gas* $\Gamma_2 = 5/3$ *and the internal energy per unit mass is* $E = (3P)/\rho$.

For a polytrope of index n, *you may assume that the central pressure and central density are given by* $P_c = c_n G M^2/R^4$, *and* $\rho_c = 3\,a_n M/(4\,\pi R^3)$ *respectively, with* c_n *and* a_n *being known constants.*

1 In a cluster of chemically homogeneous massive stars with convective cores and radiative envelopes, the stellar material is an ideal gas with radiation pressure being negligible. The energy generation rate, due to the CNO cycle, is given by $\epsilon = \epsilon_0 \rho T^{13}$, where ϵ_0 and ν are constants. The opacity is given by $\kappa = \kappa_0 \rho T^{-3}$, where κ_0 is a constant.

A set of dimensionless variables are defined through $x = r/R$, $q = m/M$, $l = L_r/L$, $b = (4\pi\rho R^3)/M$, $p = (4\pi R^4 P)/(GM^2)$ and $t = \mathcal{R}TR/(\mu GM)$ with q, l, b, p, t being functions only of x.

Show that in terms of these variables, the equations of stellar structure take the form

$$\begin{aligned}
\frac{dq}{dx} &= x^2 b\,, \\
\frac{dp}{dx} &= -\frac{bq}{x^2}\,, \\
\frac{dt}{dx} &= -\min\left(\frac{2q}{5x^2}\,,\, D\,\frac{b^2 l}{x^2 t^6}\right)\,, \\
\frac{dl}{dx} &= E\,x^2\,b^2\,t^{13}\,.
\end{aligned}$$

Here $\min(F, G)$ denotes the minimum of F and G and the constants D and E are given by

$$D = \frac{3\,\kappa_0 L}{256\,\pi^3 acM^5}\left(\frac{\mathcal{R}}{\mu G}\right)^7 \qquad \text{and} \qquad E = \frac{\epsilon_0 M^{15}}{4\,\pi L R^{16}}\left(\frac{\mu G}{\mathcal{R}}\right)^{13}.$$

State the boundary conditions to be satisfied by p, t, l, q and show that the group of stars obeys the mass-luminosity and mass-radius relations

$$L \propto \mu^7 M^5 \qquad \text{and} \qquad R \propto \mu^{3/8} M^{5/8}\,.$$

Retaining the dependence on μ, find the form of $\log L - \log T_e$ relation (HR diagram).

These stars evolve away from the main sequence as hydrogen is converted to helium. Assuming the composition remains homogeneous and adopting the approximation $\mu = 4/(3 + 5X)$, show that the luminosity of a star is given as a function of time, t, by

$$L = L_0\left(1 - \frac{40 L_0 t}{M E_H (3 + 5X_0)}\right)^{-7/8},$$

where E_H is the energy liberated per unit mass when hydrogen is converted to helium and L_0 and X_0 are the initial values (at $t = 0$) of L and X respectively.

2 Derive the virial theorem for a spherically symmetric star in hydrostatic equilibrium with zero boundary pressure in the form

$$3\int_0^M \frac{P}{\rho}\, dm + \Omega = 0\,,$$

where Ω is the gravitational energy.

A star with a fully convective interior, consisting of ideal gas with no internal energy sources, is contracting towards the main sequence. Radiation pressure is negligible. Show that

$$\frac{dR}{dt} = \frac{-7LR^2}{3GM^2}\,.$$

The temperature in the upper atmospheric radiative layers is given as a function of the optical depth τ by

$$T^4 = T_e^4 \left(\frac{1}{2} + \frac{3}{4}\tau\right)$$

and the opacity is given by $\kappa = \kappa_0 \rho T^{17}$, where κ_0 is constant. Show that in the atmosphere

$$P^2 = \frac{512\,\mathcal{R}g}{9\mu\,\kappa_0 T_e^{16}}\left[\frac{1}{8} - \frac{1}{(2+3\tau)^3}\right],$$

where the surface gravity $g = GM/R^2$.

Given the condition for instability to convection in the form

$$\frac{P}{T}\frac{dT}{dP} > \frac{2}{5}\,,$$

deduce that the onset of convection occurs when $T = (17/40)^{1/12}\, T_e$. Assuming that a relation of the form $P = KT^{5/2}$, with K being constant, applies throughout the convection zone, show that there is a relation between the mass, radius and luminosity of the form

$$L \propto M^{8/21} R^{46/21}\,.$$

Hence deduce that if the star starts from a very large radius at $t = 0$, then at later times

$$L = \frac{9\,GM^2}{67Rt}\,.$$

(You may assume that the gravitational energy of a polytrope of index n is given by $\Omega = -3\,GM^2/((5-n)R)$.)

3 a) Show that the electron pressure in a helium gas in which the electrons are completely degenerate and highly relativistic is given by

$$P = K\rho^{4/3},$$

where

$$K = \left(\frac{3}{2\pi}\right)^{1/3} \frac{hc}{16 m_p^{4/3}},$$

with h, c, and m_p being Planck's constant, the speed of light and the mass of a proton respectively.

Deduce that in this limit there is one possible mass for a white dwarf given by

$$M = \left(\frac{3}{2\pi}\right)^{1/2} \left(\frac{3a_3}{4\pi}\right)^{2} \left(\frac{1}{64 c_3^{3/2}}\right) \left(\frac{hc}{G}\right)^{3/2} \frac{1}{m_p^2}.$$

Briefly explain the significance of this result for stellar evolution.

(You may assume that for complete degeneracy, the number density of electrons, $n(p)$, with total momentum less than p, is given by $dn(p)/dp = 8\pi p^2/h^3$, $p < p_0$ and $dn(p)/dp = 0$, $p > p_0$, where p_0 is the Fermi momentum.)

b) The degenerate core of a red giant has a mass M_c and radius R_c.

Above the core is a hydrogen rich radiative envelope with a negligible mass. Hydrogen burns in a thin shell at the base of the envelope where the entire luminosity is generated. The opacity is given by $\kappa = \kappa_0$ where κ_0 is a constant. The equation of state for the envelope is that of an ideal gas plus radiation, with the contribution of the radiation pressure being taken into account.

Assuming the radiative envelope extends to small values of P and T, show that in the regions above the shell

$$P = \Lambda T^4 \,,$$

where the constant

$$\Lambda = \frac{a}{3(1-\beta)} = \frac{4\,\pi ac\,GM_c}{3\kappa_0 L} \,,$$

with β being the ratio of the gas pressure to the total pressure.

Show further that T as a function of r is given by

$$T = \frac{\mu\beta GM_c}{4\,\mathcal{R}r} \,.$$

The energy generation rate in the hydrogen rich layers is given by $\epsilon = \epsilon_0\,\rho T^{16}$.

Deduce that the luminosity is given by an expression of the form

$$L = C\,\frac{(\mu\beta M_c)^8}{R_c^{19/3}} \,,$$

where C depends only on κ_0, ϵ_0 and physical constants.

4 A binary system has components of mass M_1 and M_2 in circular orbit about their centre of mass with period $P_{orb} = 2\pi/\Omega$. The orbital separation is a.

Show that the orbital angular momentum is given by

$$J = \left(\frac{M_1 M_2}{M_1 + M_2}\right) a^2 \Omega = \frac{G^{2/3} P_{orb}^{1/3} M_1 M_2}{(2\pi)^{1/3} (M_1 + M_2)^{1/3}} .$$

The star of mass M_1 transfers mass to M_2 and loses mass to infinity through a stellar wind. The mass transfer rate is $\dot{M}_2 = -f\dot{M}_1$ and the mass loss rate due to the wind is $(1-f)\dot{M}_1$. The wind carries away a specific angular momentum equal to that of the binary multiplied by a constant factor λ. Conservation of angular momentum then gives $\dot{J} = \lambda(1-f)J\dot{M}_1/(M_1+M_2)$.

Show that during this evolution, the orbital period satisfies the relation

$$P_{orb} \propto M_1^{-3} M_2^{-3} (M_1 + M_2)^{\delta} ,$$

where $\delta = 3\lambda + 1$.

The radius, R, of M_1 expands as a result of stellar evolution obeying the relation

$$\ln R = n \ln M_1 + t/t_{nuc} ,$$

where t_{nuc} is a constant nuclear time scale and n is a constant.

The radius of the Roche lobe of M_1 is given by the relation

$$R_L = 0.46a \left(\frac{M_1}{(M_1 + M_2)}\right)^{1/3} .$$

Suppose that as a result of stellar evolution, R slightly exceeds R_L and that as a result $\dot{M}_2$ is given by

$$\dot{M}_2 = (M_1 x)/t_{dyn} ,$$

where $x = \ln(R/R_L)$ and t_{dyn} is a constant dynamical time scale.

Deduce that

$$\frac{dx}{dt} = \frac{1}{t_{nuc}} - \frac{x}{f t_{dyn}} \left(n + \frac{5}{3} - 2fq - \frac{2\,\delta(1-f)q}{3(1+q)}\right) .$$

Hence find a condition on $q = M_1/M_2$ that is required to ensure that mass transfer can proceed slowly on a nuclear time scale and indicate briefly what you would expect to happen if it is not satisfied.

END OF PAPER