MATHEMATICAL TRIPOS Part III

Friday 30 May 2008 9.00 to 12.00

PAPER 7

COMMUTATIVE ALGEBRA

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

In the following questions, A always denotes a commutative ring with unit element. All rings are tacitly assumed to be commutative and possess a unit element. Results presented in the lectures can be used without proof - unless you are explicitly asked to give a proof - but their use should be properly indicated. Results from examples sheets should not be used without proof.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let $\mathfrak{a} \subset A$ be an ideal and $\mathfrak{m} \subset A$ a prime ideal. We consider the quotient $B = A/\mathfrak{a}$ as an A-module in the obvious way. Show that the localisation $B_\mathfrak{m}$ of B is the zero module if and only if $\mathfrak{a} \not\subset \mathfrak{m}$.

(b) Let M be an A-module. Show that M is the zero module if and only if for all maximal ideals $\mathfrak{m} \subset A$ the localisation $M_\mathfrak{m}$ is the zero module. (It may be helpful to consider a submodule $Ax \subset M$ generated by one element, and apply part (a).)

(c) Let $f : M \to N$ be an A-module homomorphism. Show that f is injective if and only if for all maximal ideals $\mathfrak{m} \subset A$ the induced homomorphism $f_\mathfrak{m} : M_\mathfrak{m} \to N_\mathfrak{m}$ between the localisations is injective.

(d) Let E be an A-module. Show that E is flat if and only if for all maximal ideals $\mathfrak{m} \subset A$ the localisation $E_\mathfrak{m}$ is a flat $A_\mathfrak{m}$-module.

2 (a) Give an example of a ring B, a B-module N and an exact sequence of B-modules

$$0 \to M' \to M \to M'' \to 0$$

such that the induced sequence

$$0 \to \mathrm{Hom}_B(N, M') \to \mathrm{Hom}_B(N, M) \to \mathrm{Hom}_B(N, M'') \to 0$$

is not exact.

(b) Let F be a flat A-module, and $P, Q \subset F$ be two submodules such that $F = P \oplus Q$. Show that P is a flat A-module.

(c) An A-module P is *projective* if for every exact sequence

$$0 \to M' \to M \to M'' \to 0$$

the induced sequence

$$0 \to \mathrm{Hom}_B(P, M') \to \mathrm{Hom}_B(P, M) \to \mathrm{Hom}_B(P, M'') \to 0$$

is exact. Show that a projective A-module is a direct summand of a free A-module. (You may first want to show that every module is the quotient of a free module.)

(d) Let P be a projective A-module. Show that P is flat.

3 In the following $(A, \mathfrak{m})$ is a local Noetherian ring with residue field $k = A/\mathfrak{m}$. We put $d = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$.

(a) Show that $\mathfrak{m}$ can be generated by d elements. In particular, the least number of generators of an $\mathfrak{m}$-primary ideal is less or equal to d.

(b) Prove that the ring of formal power series $B = A[[x]]$ is a local ring with maximal ideal $I = \mathfrak{m}B + xB$ and residue field k. Show that $\dim_k(I/I^2) = d + 1$ and $\dim(B) \leq d+1$. (When quoting a result from the lectures you may assume without proof that B is Noetherian.)

(c) Now assume that A is a regular local ring, i.e. $d = \dim(A)$. Let $B = A[[x]]$ be as above. Prove that $\dim(B) \geq d+1$. (One may consider a chain of prime ideals in A, and then construct a suitable chain of prime ideals in B.) Hence conclude that $\dim(B) = d+1$.

4 (a) Let $\phi : A \to B$ be a ring homomorphism and $\mathfrak{p} \subset A$ be a prime ideal of A. Show that the set of prime ideals $\mathfrak{q}$ of B with $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ is in canonical bijection with $\mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$, where $\kappa(\mathfrak{p})$ is the field of fractions of $A/\mathfrak{p}$.

(b) Let K be a field and C a K-algebra which is finite-dimensional as K-vector space. Proof that every prime ideal of C is maximal and that C has only finitely many maximal ideals.

(c) Let $A \subset B$ be an integral ring extension, and suppose B is finitely generated as A-algebra. Show that for every prime ideal $\mathfrak{p} \subset A$ there are only finitely many prime ideals $\mathfrak{q}$ of B with $A \cap \mathfrak{q} = \mathfrak{p}$

END OF PAPER