UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 5 June 2008 9.00 to 12.00

PAPER 69

ASTROPHYSICAL FLUID DYNAMICS

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.
You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\rho = -\rho\boldsymbol{\nabla}\cdot\mathbf{u}\,,$$

$$\frac{\partial p}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}p = -\gamma p\boldsymbol{\nabla}\cdot\mathbf{u}\,,$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right) = -\rho\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}p + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}\,,$$

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B})\,,$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0\,,$$

$$\nabla^2\Phi = 4\pi G\rho\,.$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Consider axisymmetric but time-dependent solutions of ideal MHD. Show that the poloidal part of the magnetic field can be described using a scalar magnetic flux function. Describe the properties of this flux function and show that it is conserved following the fluid motion. Is the toroidal part of the magnetic field conserved in any sense?

(b) Let $z_\pm = z \pm v(R)t$, where (R, ϕ, z) are cylindrical polar coordinates. Under what conditions on the functions f, g and v is

$$\begin{aligned}\mathbf{u} &= [f(R, z_+) + g(R, z_-)]\,\mathbf{e}_\phi\,,\\ \mathbf{B} &= (\mu_0\rho)^{1/2}\left\{[f(R, z_+) - g(R, z_-)]\,\mathbf{e}_\phi + v(R)\,\mathbf{e}_z\right\}\,,\\ \rho &= \text{constant}\end{aligned}$$

a solution of the equations of ideal MHD in an incompressible fluid? Give a detailed physical interpretation of the available solutions of this type.

2 (a) Show that the equation of motion for the adiabatic flow of an unmagnetized fluid can be written in the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla(\tfrac{1}{2}u^2 + \Phi + w) = T\nabla s\,,$$

where w, T and s should be defined.

(b) The wind from a rotating star can be modelled as a steady, axisymmetric, adiabatic flow in which the magnetic field is neglected. Let $\psi(R, z)$ be the mass flux function, such that

$$\rho\mathbf{u}_\mathrm{p} = \nabla\psi \times \nabla\phi\,,$$

where (R, ϕ, z) are cylindrical polar coordinates and $\mathbf{u}_\mathrm{p}$ is the poloidal part of $\mathbf{u}$. Show that the specific entropy, the specific angular momentum and the Bernoulli function are constant along streamlines, giving rise to three functions $s(\psi)$, $\ell(\psi)$ and $\varepsilon(\psi)$.

Use the remaining dynamical equation to show that ψ satisfies the partial differential equation

$$\frac{1}{\rho}\nabla\cdot\left(\frac{1}{\rho R^2}\nabla\psi\right) = \frac{\mathrm{d}\varepsilon}{\mathrm{d}\psi} - T\frac{\mathrm{d}s}{\mathrm{d}\psi} - \frac{\ell}{R^2}\frac{\mathrm{d}\ell}{\mathrm{d}\psi}\,.$$

3 An isothermal gas satisfies $p = c_\mathrm{s}^2\rho$, where c_s is a constant. Let (x, y, z) be Cartesian coordinates, and consider a flow such that the fluid variables depend only on x and t, while the z-components of $\mathbf{u}$ and $\mathbf{B}$ vanish. Show that, if gravity may be neglected, the equations of ideal MHD can be written in the form

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} = \mathbf{0}\,,$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ u_x \\ u_y \\ B_y \end{bmatrix}$$

and $\mathbf{A}$ is a 4×4 matrix to be determined, while B_x is a constant.

Show that the eigenvalues v of $\mathbf{A}$ satisfy

$$(v - u_x)^4 - (c_\mathrm{s}^2 + v_\mathrm{a}^2)(v - u_x)^2 + c_\mathrm{s}^2 v_{\mathrm{a}x}^2 = 0\,,$$

where $\mathbf{v}_\mathrm{a}$ is the Alfvén velocity.

Explain the significance of these results. Show that the equations admit solutions in the form of simple nonlinear waves that undergo wave steepening.

4 Consider small-amplitude, purely radial oscillations of a spherically symmetric star. Show that the radial displacement $\xi_r(r, t)$ satisfies the equation

$$\frac{\partial^2 \xi_r}{\partial t^2} = \frac{1}{\rho r^3}\frac{\partial}{\partial r}\left[\gamma p r^4 \frac{\partial}{\partial r}\left(\frac{\xi_r}{r}\right)\right] - (3\gamma - 4)\frac{g\xi_r}{r}\,,$$

where g is the inward gravitational acceleration and γ may be assumed to be constant.

Deduce the expression

$$\omega^2 \int_0^R \rho r^2 |\xi_r|^2 \,\mathrm{d}r = \int_0^R \left[\gamma p r^4 \left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\xi_r}{r}\right)\right|^2 + (3\gamma - 4)\rho g r |\xi_r|^2\right] \mathrm{d}r$$

for the frequencies ω of oscillation modes of the star, where R is the stellar radius. (Only a very brief discussion of the boundary conditions is required.)

Assuming that this expression has the usual variational property, use a suitable trial function to obtain the upper bound

$$\omega_\mathrm{min}^2 \leq (3\gamma - 4)\frac{4\pi}{3}G\rho_\mathrm{c}$$

on the lowest oscillation frequency in the case $\gamma > 4/3$, where ρ_c is the central (and maximum) density of the star. What happens if $\gamma < 4/3$?

END OF PAPER