UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2008 1.30 to 4.30

PAPER 66

BLACK HOLES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State (without explanation) which of the space-times listed below is (i) extendible; (ii) geodesically complete; (iii) globally hyperbolic; (iv) asymptotically simple. (Present your answers in the form of a table.)

The region $r > 2M$ of the Schwarzschild solution (with $M > 0$),

Kruskal,

the maximal analytic extension of extremal Reissner-Nordstrom,

Rindler,

the Einstein static universe.

(b) The Majumdar-Papapetrou (MP) solution has metric

$$ds^2 = -H^{-2}dt^2 + H^2\left(dx^2 + dy^2 + dz^2\right) , \qquad H(x,y,z) = 1 + \sum_{i=1}^{N} \frac{M_i}{|\mathbf{x} - \mathbf{x}_i|}$$

where $M_i > 0$, $\mathbf{x} = (x, y, z)$, and $\mathbf{x}_i$ are constant vectors in $\mathbb{R}^3$.

(i) Define the *Komar mass*, and calculate it for the MP solution.

(ii) Without loss of generality, take $\mathbf{x}_1 = 0$. Let (r, θ, ϕ) denote spherical polar coordinates in $\mathbb{R}^3$. Show that the MP solution can be analytically extended through a Killing horizon at $r = 0$, and calculate the surface gravity of this horizon.

(*Hint.* Define new coordinates (v, r, θ, ϕ) where $dv = dt + (h + M_1/r)^2\, dr$ for a suitably chosen constant h.)

2

Consider a null geodesic congruence. Let U be tangent to the (affinely parametrized) null geodesics.

(a) What is a *deviation vector*?

(b) Define the *expansion*, θ, *shear*, $\hat{\sigma}$, and *twist*, $\hat{\omega}$, of the congruence in terms of a matrix $\hat{B}^a{}_b$ that you should define.

(i) Derive an equation relating $U \cdot \nabla \hat{B}^a{}_b$ to the expansion, shear, twist, and Riemann tensor. Hence, or otherwise,

(ii) obtain Raychaudhuri's equation;

(iii) show that

$$U \cdot \nabla \hat{\omega}_{ab} = -\theta\, \hat{\omega}_{ab}\,, \qquad U \cdot \nabla \hat{\sigma}_{ab} = -\theta\, \hat{\sigma}_{ab} - P^e_a\, P^f_b\, C_{cedf}\, U^c U^d\,,$$

where P^a_b should be defined, and C_{abcd} is the Weyl tensor, defined by

$$C_{abcd} = R_{abcd} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a} + (1/3)R\, g_{a[c}g_{d]b}\,.$$

(*Hint.* At any point, $\hat{\sigma}$ and $\hat{\omega}$ can be regarded as 2×2 matrices acting on a vector space $T_\perp$ with metric P_{ab}. One can introduce an orthonormal basis for $T_\perp$, with respect to which P_{ab} is the identity matrix.)

3

(a) State and prove the version of the first law of black hole mechanics that relates the change in area of the event horizon to the energy and angular momentum of infalling matter. (You may assume Raychaudhuri's equation.)

(b) Consider the Penrose process for a Kerr black hole. By considering the 4-momentum, explain why the particle that falls through the horizon has energy E and angular momentum L obeying $E \geq \Omega_H L$. Show that the same result can be obtained from the first and second laws of black hole mechanics.

4

Consider spherically symmetric gravitational collapse to form a Schwarzschild black hole. Write an essay explaining why, at late times, an observer at infinity will see thermal radiation coming from the black hole.

END OF PAPER