UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 9 June 2008 1.30 to 3.30

PAPER 64

ADVANCED COSMOLOGY

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) In a flat FRW universe ($\Omega = 1$) assume that the matter content can be described as a perfect fluid with energy-momentum tensor

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}\,,$$

where ρ is the energy density, P is the pressure (satisfying the equation of state $P = w\rho$) and $u^\mu \approx a^{-1}(1,\,\mathbf{v})$ is the four-velocity of the fluid. In synchronous gauge (with perturbed line element $ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right]$), show that the linearized energy-momentum tensor can be written in the form

$$T^{00} = \frac{1}{a^2}\bar\rho(1+\delta)\,, \qquad T^{0i} = \frac{1}{a^2}\bar\rho(1+w)v^i\,, \qquad T^{ij} = \frac{1}{a^2}\bar\rho w[(1+\delta)\delta_{ij} - h_{ij}]\,,$$

where δ is the density perturbation and $\bar\rho$ is the homogeneous background density. Show that energy-momentum conservation implies that the perturbations obey the following equations

$$\begin{aligned} &\delta' + (1+w)i\mathbf{k}\cdot\mathbf{v} + \tfrac{1}{2}(1+w)h' = 0\,,\\ &\mathbf{v}' + (1-3w)\frac{a'}{a}\mathbf{v} + \frac{w}{1+w}i\mathbf{k}\delta = 0\,, \end{aligned} \qquad (\dagger)$$

where $\mathbf{k}$ is the comoving wavevector ($k = |\mathbf{k}|$) and primes denote differentiation with respect to conformal time τ ($d\tau = dt/a$).

[*Hint:* You may assume that $\Gamma^0_{00} = \frac{a'}{a}$, $\Gamma^0_{0i} = \Gamma^i_{00} = 0$, $\Gamma^0_{ij} = \frac{a'}{a}(\delta_{ij} + h_{ij}) + \frac{1}{2}h'_{ij}$, $\Gamma^i_{0j} = \frac{a'}{a}\delta_{ij} + \frac{1}{2}h'_{ij}$ and $\Gamma^i_{jk} = \frac{1}{2}(h_{ij,k} + h_{ik,j} - h_{jk,i})$.]

(ii) Now assume that the late universe is dominated by a non-relativistic fluid component $\rho_{\rm m}$ well after matter-radiation equality at $t_{\rm eq}$ and that you are given the scalar trace metric perturbation equation ($h \equiv h_{ii}$):

$$h'' + \frac{a'}{a}h' + 3\left(\frac{a'}{a}\right)^2(1+3w)\delta_{\rm m} = 0\,.$$

Show from this equation, together with ($\dagger$) that if the non-relativistic pressure satisfies $P_{\rm m} = w_m\rho_{\rm m} \ll \rho_m$ (with w_m const.), then the density perturbation $\delta_{\rm m}$ will obey:

$$\delta''_m + \frac{a'}{a}\delta'_m - \left[4\pi G\bar\rho_m a^2 - c_s^2 k^2\right]\delta_m = 0\,, \qquad (\ddagger)$$

where the sound speed is $c_{\rm s}^2 \equiv dP/d\rho$, here with $w_m = c_s^2$. Define the Jeans length $\lambda_{\rm J}$ and briefly discuss its importance for structure formation before and after recombination.

Define the variance σ_R of a perturbation on a specific physical lengthscale R. For an initial power spectrum $P(k) = Ak$ at $t = t_{\rm eq}$ in the non-relativistic matter perturbations $\delta_{\rm m}$ which obey ($\ddagger$), show that the variance is constant at horizon crossing $k \sim aH$ (i.e. the perturbations are scale-invariant).

2 A photon with four-momentum p^μ ($p_\mu p^\mu = 0$) propagating in a flat ($\Omega = 1$) but perturbed FRW universe with line element

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right] ,$$

at linear order obeys

$$\frac{dq}{d\tau} = -\frac{1}{2} q h'_{ij}\hat{n}^i\hat{n}^j , \qquad \frac{d\hat{n}^i}{d\tau} = \mathcal{O}(h_{ij}) . \qquad (*)$$

where q is the comoving photon momentum, $\hat{n}^i$ is the (unit) photon propagation direction and primes denote derivatives with respect to conformal time τ.

(i) The photon distribution function $f(\mathbf{x}, \mathbf{p}, \tau)$ can be expanded about the Planck spectrum $f_0(p,\tau) = f_0(q)$ as

$$f(\mathbf{x}, \mathbf{p}, \tau) = f_0(q) + f_1(\mathbf{x}, q, \hat{\mathbf{n}}, \tau) ,$$

where the photon momentum $p \equiv q/a$. Show that the collisionless Boltzmann equation

$$\frac{df}{d\lambda} \equiv \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} + \frac{dp^\mu}{d\lambda}\frac{\partial f}{\partial p^\mu} = 0$$

can be re-expressed in the form

$$\frac{\partial f_1}{\partial \tau} + \hat{n}^i \frac{\partial f_1}{\partial x^i} + \frac{dq}{d\tau}\frac{df_0}{dq} + \frac{dq}{d\tau}\frac{\partial f_1}{\partial q} + \frac{d\hat{n}^i}{d\tau}\frac{\partial f_1}{\partial \hat{n}^i} = 0 ,$$

which, using the results from ($*$), at linear order reduces to

$$\frac{\partial f_1}{\partial \tau} + ik\mu f_1 = \frac{1}{2} q \frac{df_0}{dq} h'_{ij}\hat{n}^i\hat{n}^j ,$$

where $\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}$. Finally, argue that the brightness function

$$\Delta(\mathbf{x}, \hat{\mathbf{n}}, \tau) \equiv 4\frac{\Delta T}{T} \equiv \frac{4\pi}{a^4 \rho_\gamma}\int q f_1 q^2 dq$$

must therefore satisfy

$$\Delta' + ik\mu\Delta = -2h'_{ij}\hat{n}^i\hat{n}^j . \qquad (\dagger)$$

(ii) Argue that if the photon fluid is in equilibrium for $\tau \leq \tau_{\rm dec}$, we may approximate the initial conditions for the photon brightness at decoupling by

$$\Delta(\mathbf{k},\, \mu,\, \tau_{\rm dec}) = \delta_\gamma(\tau_{\rm dec}) + 4\hat{\mathbf{n}} \cdot \mathbf{v}(\tau_{\rm dec}) ,$$

that is, briefly justify why the higher order moments $\Delta_\ell \approx 0$ ($\ell \geq 2$) can be neglected. Hence, assuming instantaneous decoupling, integrate ($\dagger$) from decoupling $\tau_{\rm dec}$ to today τ_0 to find the Sachs-Wolfe formula for the CMB temperature anisotropy seen at position $\mathbf{x}$ in a direction $\mathbf{n}$:

$$\frac{\Delta T}{T}(\mathbf{x}, \mathbf{n}, \tau_0) = \frac{1}{4}\delta_\gamma(\mathbf{x}, \tau_{\rm dec}) + \hat{\mathbf{n}} \cdot \mathbf{v}(\mathbf{x}, \tau_{\rm dec}) - \frac{1}{2}\int_{\tau_{\rm dec}}^{\tau_0} d\tau h'_{ij}\, n^i n^j . \qquad (\ddagger)$$

Briefly explain the meaning of each term in the formula ($\ddagger$), and describe their scale dependence on large and small angles.

3 Consider slicing spacetime into constant time t hypersurfaces Σ_t each with three metric ${}^{(3)}g_{ij}(x^i)$. The proper distance between two points on Σ_t and Σ_{t+dt} can then be expressed as

$$ds^2 = -N^2 dt^2 + {}^{(3)}g_{ij}(dx^i - N^i dt)(dx^j - N^j dt)\,,$$

where the lapse function $N(t, x^i)$ defines the change in the proper time and the shift vector $N^i(t, x^i)$ the change in the spatial coordinates for a 'normal' trajectory defined along $n_\mu = (-N,\ 0,\ 0,\ 0)$. For a scalar field ϕ with Lagrangian $\sqrt{-g}[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]$ and momentum $\Pi = N^{-1}(\dot\phi + N^i\phi_{|i})$, the evolution equations in this metric become

$$\dot\Pi + N^i\Pi_{|i} - NK\Pi - N^{|i}\phi_{|i} + N\phi^{|i}_{|i} + N\tfrac{dV}{d\phi} = 0\,, \qquad \tilde{K}^j{}_{i|j} - \tfrac{2}{3}K_{|i} = -8\pi G\Pi\phi_{|i}\,,$$
$${}^{(3)}R + \tfrac{2}{3}K^2 - \tilde{K}_{ij}\tilde{K}^{ij} = 16\pi G[\tfrac{1}{2}\Pi^2 + \tfrac{1}{2}\phi_{|i}\phi^{|i} + V(\phi)]\,,$$
$$\dot K + N^iK_{,i} + N^{|i}{}_{|i} - N({}^{(3)}R + K^2) = -8\pi GN[2\phi_{|i}\phi^{|i} + 3V(\phi)]\,,$$
$$\dot{\tilde{K}}{}^i{}_j + N^k\tilde{K}^i_{j|k} - N^i{}_{|k}\tilde{K}^k_j + N^k{}_{|j}\tilde{K}^i_k + N^{|i}{}_{|j} - \tfrac{1}{3}N^{|k}{}_{|k}\delta^i{}_j$$
$$- N({}^3\tilde{R}^i_j + K\tilde{K}^i_j) = -8\pi GN[\phi^{|i}\phi_{|j} - \tfrac{1}{3}\delta^i{}_j\phi_{|k}\phi^{|k}]\,,$$

where $|$ denotes the covariant derivative in Σ, the intrinsic curvature is ${}^{(3)}R_{ij}$ (with Ricci scalar ${}^{(3)}R$) and the extrinsic curvature is K_{ij}, which splits into trace and traceless parts respectively, $K \equiv {}^{(3)}g_{ij}K^{ij}$ and $\tilde{K}_{ij} \equiv K_{ij} - \frac{1}{3}{}^{(3)}g_{ij}K$.

(i) The extrinsic curvature is given as $K_{ij} \equiv -n_{i;j} = -\frac{1}{2}N^{-1}\left({}^{(3)}g_{ij,0} + N_{i|j} + N_{j|i}\right)$. Consider the conformal 3-metric ${}^{(3)}\tilde{g}_{ij} = a^2(t,\, x^i){}^{(3)}g_{ij}$ where $a^6 \equiv {}^{(3)}g = \det({}^{(3)}g_{ij})$ and, hence or otherwise, take the trace of the extrinsic curvature expression to find

$$K \equiv {}^{(3)}g^{ij}K_{ij} = -\frac{1}{2N}\left(\frac{{}^{(3)}\dot g}{{}^{(3)}g} + 2N^i_{|i}\right)\,.$$

In the context of an expanding universe (setting $N^i = 0$), argue that $H(t, x^i) \equiv -K/3 = \dot a(t,\, x^i)/a(t,\, x^i)$ can be interpreted as a locally defined Hubble parameter. [*Hint:* You may assume that $\mathrm{Tr}(A^{-1}dA/dt) = d(\ln(\det A))/dt$ for any matrix A with $\det A \neq 0$.]

(ii) Explain the long wavelength approximation and why it is accurate under some circumstances to neglect second order gradients. Rewrite the Einstein equations in long wavelength form (again with shift $N^i = 0$).

Show that the traceless part of the extrinsic curvature has the general solution $\tilde{K}^i_j \approx C^i_j(x)\,a^{-3}$. Discuss the significance of this result for an inflationary universe. Hence, also show that

$$\dot\Pi = -\left(3H + \frac{1}{\Pi}\frac{dV}{d\phi}\right)\dot\phi\,, \qquad \Pi_{|i} = -\left(3H + \frac{1}{\Pi}\frac{dV}{d\phi}\right)\phi_{|i}\,,$$
$$\dot H = -4\pi G\Pi\,\dot\phi\,, \qquad H_{|i} = -4\pi G\Pi\,\phi_{|i}\,. \qquad (\dagger)$$

(iii) Use the long wavelength Einstein equations (†) to prove that the nonlinear inhomogeneous variable

$$\zeta_i = -\frac{\partial_i a}{a} + \frac{H}{\Pi}\partial_i\phi\,.$$

is conserved on superhorizon scales, that is, $\dot\zeta_i = 0$. Briefly discuss the implications of this result for nonGaussianity from single field inflation.

END OF PAPER